MATHEMATICS MAGAZINE

Vol. 25, No. 4, March-April, 1952

CONTENTS

"Understandable Chapters on Various Courses in Mathematics" will continue in subsequent issues.

OUR CONTRIBUTORS

W. C. Kalinowski is Associate Professor of Mathematics at St. John's University, Collegeville, Minnesota. A graduate of St. John's (B.A. '35), he did graduate work at the Universities of Wisconsin, Minnesota, and the University of St. Louis (Ph.D. '48). Professor Kalinowski's special mathematical interests lie in statistics and probability.

Francis Regan, Professor and Director, Department of Mathematics, St. Louis University, was born in Indiana in 1903. He attended Indiana State Teachers College (A.B., 1922), Indiana University (A.M., 1930) and the University of Michigan (Ph.D., 1932), where he was a University Fellow. From 1923-25, he taught in Indiana High Schools. He taught commerce and mathematics at Columbus College, Sioux Falls, S. D., (1925-29), and then became Assistant Professor of Mathematics at Colorado A. and M. College (1929-30). Dr. Regan has been at St. Louis University since 1932. He is a member of Phi Beta Kappa, Pi Mu Epsilon and Sigma Xi; and is chairman of the Missouri Section of the Mathematical Association for 1951-52. His special interests are in the fields of probability and analysis.

La Mar I. Deverall, Instructor, University of Utah, was born in Taylorville, Utah, in 1924. Mr. Deverall is a graduate of the University of Utah (B.S. '46, M.S. '48). His particular mathematical interests are Transform Methods and Differential Equations.

C. J. Thorne is Associate Professor of Mathematics at the University of Utah. A native of Utah, he attended Brigham Young University (A.B. '36), Iowa State College (M.S. '38; Ph.D. '41) and then became a post-doctoral fellow at Brown University (Summer, '42). Dr. Thorne has taught at a number of Universities, including Louisiana State, Michigan, Washington University and U.C.L.A., as well as Utah. During 1944-45 he also served as a development engineer with Curtiss-Wright. His mathematical interests are in analysis and applied mathematics.

Nicholas A. Draim, Captain, U. S. Navy, was born in Vincennes, Indiana, in 1901. Besides being an honor graduate of the U. S. Naval Academy ('22) he attended the Massachusetts Institute of Technology (M.Sc. '25) and Georgetown Law School (LLB, '34). During World War II he served in the South Pacific and received the Legion of Merit. From 1949-51 he was a Naval Attache and Naval Attache for Air, American Embassy, Moscow, U.S.S.R. Besides being an Associate Fellow, Institute of Aeronautical Sciences, Capt. Draim is also a Member of the Bar of the U. S. Court of Customs and Patent Appeals and of the U. S. Supreme Court.

(*Continued on Page 182.*)

A POSTULATIONAL TREATMENT OF THE PROBABILITY FOR CERTAIN TYPES OF EMISSIONS

Walbert C. Kalinowski and Francis Regan

1. *Introduction.* We shall present a set of postulates for the purpose of determining the probability distribution function for the emission of particles or rays by a radioactive substance. When a particle is emitted, it may be thought of as being an element of a sequence f in time. The different counters used by experimentalists to register these emissions or impulses do not count all of them. To translate this phenomena into a mathematical law another sequence g may be formed by omitting certain elements of f. There are two laws dealing with the formation of the set g from f. These laws correspond to the reaction of different types of counters. The rule in forming g is [4], [11]:[1]

Law 1. Let a be an element in f and g. The next element to be included in g is then the first element in f which follows a after a distance greater than a constant u.

Law 2. Let a be an element in f and g. The next element to be included in g is then the first element in f which follows a at a distance greater than a constant u from the preceding element in f, whether this belongs to g or not.

This new sequence g consists of the counted emissions. The distinction between the two laws has caused some misunderstanding but it has been clearly brought out by Ruark and Brammer [12].

v. Bortkiewicz [2], starting from investigations by Rutherford, Geiger and others, considered problems related to the sequence g (Law 1), finding that the distribution of the number of recorded impulses during a certain time interval was similar to the Poisson law but did have a smaller dispersion.

Gnedenko [6] has concerned himself with the distribution of lost emissions occurring in the first law especially with regard to the initial state of rest. He also obtained an expression for the average number of impulses in a given time. More recently Karbatov and Mann [7] have obtained a simpler expression for this average.

Levert and Scheen [9] working with the second law have found an expression for the distribution of the number of impulses during a given time for g. Further they as well as Kosten [8] found the average number and variance for these impulses.

Alaoglu and Smith [1] concerned themselves with problems dealing with successive transformations of these sequences. Malmquist [11] derives distribution functions for the first law and his generalized formulas can be used in treating problems referring to successive transformations.

[1] Numbers in [ ] indicate the reference given at the end of the paper.

Feller [4] gives special formulas for the two laws and outlines a method for estimating the error committed by replacing the expression for the average number of impulses or the variance by their asymptotic limit.

Fry [5] obtained certain probabilities for the sequence f. In this paper, we formulate a set of five postulates dealing with the trend in f. From these, we obtain the probabilities that there are no points (impulses) in an interval α beginning at an arbitrary origin; that there are n points in α at time zero; and that there are n points in α when time is not zero. These results are the same as obtained by Fry [5]. However, in 4, the probability that there are n elements in a finite set of non-abutting intervals is obtained. From these probabilities, the probability of n rays being emitted by a radioactive substance when the counter is closed, as in Law 1 is indicated for two and k closures of the counter.

2. *Postulates.* We shall set up a mathematical system dealing with the sequence f described above. This sequence of emissions is represented by a set of points on the positive time axis, which will be referred to as a time series. With a given quantity of radioactive substance, there is at any time a certain number of untransmuted particles. Let N be that number. If α is the interval, $t \leq x < \alpha + t$ on the time axis, then there is a definite probability that there will be n points of the sequence in α. Let this probability be represented by $g(n,\alpha,t,N)$, where n is the number of points in the interval α, with N untransmuted particles available at time t. Then $g(n,\alpha,t,N)$ is greater than zero whenever the interval is finite and positive and $n \leq N$. It is only reasonable to assume that the probability of no points with $\alpha = 0$ for any t and N is one; that is, $g(0,0,t,N) = 1$. It is certain that there will be no points or at least one point of the time series in the interval α. Hence $\sum\limits_{n=0}^{N} g(n,\alpha,t,N) = 1$.

Let $G(n,\alpha,t,N)$ be the probability of at least n points lying in α, then $G(n,\alpha,t,N) = \sum\limits_{i=n}^{N} g(i,\alpha,t,N)$. We shall assume further that the chance of exactly one point lying in α, where α is an infinitesimal, is proportional to the source present at the time t. Therefore, we assume that $\lim\limits_{\alpha\to 0} \dfrac{g(1,\alpha,t,N)}{\alpha} = k\cdot N$, where k is a positive constant not zero.

We are assuming that the points of the time series are distributed in such a manner as to be dependent upon the source N present at the beginning instant of α. Then the probability of the simultaneous occurrence of n_1 points in α_1 beginning at the instant t_1 and n_2 points in α_2 beginning at t_2 is determinable only when the sources present at the instants t_1 and t_2 are known. If α_1 and α_2 are abutting or non-

abutting, non-overlapping intervals, then the probability of the conjunction of these events is the product $g(n_1, a_1, t_1, N_1) g(n_2, a_2, t_2, N_2)$. It follows that $g(n, a_1 + a_2, t_1, N) = \sum_{i=0}^{n} g(n-i, a_1, t_1, N_1) g(i, a_2, t_2, N-n+i)$, when a_1 and a_2 are abutting intervals.

It is reasonable to expect that the probability of obtaining two or more points of the series in a is an infinitesimal of higher order than the probability of obtaining one or more points of the series in a. Thus we shall assume that $\lim\limits_{a \to 0} \dfrac{G(2, a, t, N)}{G(1, a, t, N)} = 0$. From this limit together with $g(1, a, t, N) = k \cdot N \cdot a + a\epsilon$, we can conclude that $\lim\limits_{a \to 0} G(1, a, t, N) = 0$.

The properties of the time series which we have described are thus expressed in abbreviated form by the following system of postulates [3].

(A) $\quad 0 < g(n, a, t, N)$, if $0 < a < \infty$, and $g(0, 0, t, N) = 1$,

for $0 \leq t < \infty$.

(B) $$\sum_{n=0}^{N} g(n, a, t, N) = 1.$$

(C) $$\lim_{a \to 0} \frac{g(1, a, t, N)}{a} = k \cdot N, \text{ where } k > 0.$$

(D) $$g(n, a_1 + a_2, t_1, N) = \sum_{i=0}^{n} g(n-i, a_1, t_1, N) g(i, a_2, t_2, N-n+i),$$

if a_1 and a_2 are abutting intervals.

(E) $$\lim_{a \to 0} \frac{G(2, a, t, N)}{G(1, a, t, N)} = 0, \text{ where } G(n, a, t, N) = \sum_{i=1}^{N} g(i, a, t, N).$$ [1]

3. *Probability of n points occurring in a.* Using the postulational system, $g(n, a, t, N)$ when n and t are zero, is determined first in the following manner. If the interval a be lengthened by da, then from Postulate (D) we have

(1) $$g(0, a + da, 0, N) = g(0, a, 0, N) \cdot g(0, da, 0, N).$$

Applying (B) to the second factor of the right side of (1), it is easily seen with the aid of Postulates (C) and (E) that we get the differential equation

$$g'(0, a, 0, N) = -k \cdot N \cdot g(0, a, 0, N),$$

[1] These assumptions do not lead to a form of the Poisson Law, due to Postulate (C). If $k \cdot N$ in (C) is set equal to a positive constant, then these assumptions would be essentially those used in [3].

the solution of which is

$$g(0,a,0,N) = Ke^{-kNa}.^{1} \tag{2}$$

When t is zero and $n > 1$, we may use the same procedure and obtain the differential equation

$$g'(n,a,0,N) = -k\cdot(N - n)\cdot g(n,a,0,N) + k\cdot(N - n + 1)\cdot g(n - 1,a,0,N). \tag{3}$$

The solution of (3) is

$$g(n,a,0,N) = \exp[-k(N - n)a]\{\int\exp k(N - n)a]\cdot k(N - n + 1)\cdot g(n - 1,a,0,N)\cdot da + K\}. \tag{4}$$

An explicit solution by induction may be obtained. Let us assume that

$$g(n - 1,a,0,N) = {}_NC_{n-1}\exp[-k\cdot N\cdot a](\exp[k\cdot a] - 1)^{n-1}$$

which is true for $n - 1 = 0$. Then (4) becomes

$$g(n,a,0,N) = \exp[-k(N - n)a]\{{}_NC_n(1 - \exp[-k\cdot a])^n + K\}. \tag{5}$$

Since $g(0,0,0,N)$ is one, it follows from (B) that $g(n,0,0,N) = 0$ if $n > 0$, whence K is zero.

From the principle of alternative compound probabilities, we see that

$$g(n,a,t,N) = {}_NC_n\left[\frac{1 - \exp[-k\cdot a]}{\exp[k\cdot t]}\right]^n\cdot\left[1 - \frac{1 - \exp[-k\cdot a]}{\exp[k\cdot t]}\right]^{N-n}. \tag{6}$$

4. *Probability of n points occurring in non-abutting intervals.* We shall first consider the case of two non-abutting intervals, a_1 and a_2. The interval a_1 is preceded by the interval t_1, while t_2 is the interval between a_1 and a_2. Let the points of the series that occur in t_1 be denoted by m_1 and those in t_2 by m_2. If n_1 points occur in a_1 and n_2 in a_2, then the probability of this joint occurrence is

$$\sum_{m_1=0}^{N-n_1-n_2} {}_NC_{m_1}\exp[-K\cdot N\cdot t_1]\cdot(\exp[K\cdot t_1] - 1)^{\,1}\cdot{}_{N-m_1}C_{n_1}\exp[-K(N - m_1)a_1]\cdot(\exp[Ka_1] - 1)^{n_1}. \tag{8}$$

$$\cdot\sum_{m_2=0}^{N-n_1-n_2-m_1} {}_{N-m_1-n_1}C_{m_2}\cdot\exp[-K(N - m_1 - n_1)t_2]\cdot(\exp[Kt_2]$$

[1]The constant K is readily shown to be one, for when a approaches zero, we shall define the $\lim_{a\to 0} g(0,a,0,N)$ to be $g(0,0,0,N)$, which is one by (A).

$$- 1)^{m_2} \cdot {}_{N-m_1-n_1-m_2}C_{n_2} \exp[-K(N - m_1 - n_1 - m_2)\alpha_2] \cdot (\exp[K\alpha_2] - 1)^{n_2}.$$

We may simplify (8) by using repeatedly the combinatorial identity ${}_NC_m \cdot {}_{N-m}C_n = {}_NC_m \cdot {}_{N-n}C_m$ and by proper rearrangements the summation signs may be replaced by equivalent binomial expressions. On this simplified expression apply the principle of compound alternative probabilities, where n_1 and n_2 can take all values from zero to n and $n_1 + n_2 = n$. When certain algebraic simplifications are performed, we find that the probability of n points occurring in the two non-abutting intervals, α_1 and α_2 is

$$(9) \qquad {}_NC_n\left\{\frac{1 - \exp[-K\alpha_1]}{\exp[Kt_1]} + \frac{1 - \exp[-K\alpha_2]}{\exp[K(t_1 + \alpha_1 + \alpha_2)]}\right\}^n \cdot \left\{1 - \frac{1 - \exp[-K\alpha_1]}{\exp[Kt_1]} - \frac{1 - \exp[-K\alpha_2]}{\exp[K(t_1 + \alpha_1 + \alpha_2)]}\right\}^{N-n}.$$

To obtain the probability of n points occurring in the union of s non-abutting intervals, a procedure similar to the above is used. Let n_i points occur in α_i, m_i points occur in t_i and $\sum_{i=1}^{s} n_i = n$. After certain manipulations, we find that the probability of exactly n_i points lying in $\alpha_i (i = 1, 2, \ldots, s)$, is

$$(10) \qquad {}_NC_n\left\{\frac{n!}{n_1! \ldots n_s!} \exp[-KN(\sum_{i=1}^{s} \alpha + \sum_{i=1}^{s} t_i)] \cdot (\exp[K\alpha_s] - 1)^{n_s} (\exp[K(\alpha_{s-1} + \alpha_s + t_s)] - \exp[Kt_s])^{n_{s-1}} \ldots (\exp[k(\sum_{i=1}^{s} \alpha_i + \sum_{i=2}^{s} t_i)] - \exp[k\{\sum_{i=2}^{s} (\alpha_i + t_i)\}])^{n_1}\right\} \cdot \left\{1 - \exp[K\alpha_s] + \exp[K(\alpha_s + t_s)] - \exp[k(\alpha_{s-1} + \alpha_s + t_s] + \cdots - \exp[k(\sum_{i=1}^{s} \alpha_i + \sum_{i=2}^{s} t_i)] + \exp[k\{\sum_{i=1}^{s} (\alpha_i + t_i)\}]\right\}^{N-n}.$$

The expression in brackets is the general term of a multinomial. Therefore, when we sum up the terms allowing each of the n_i to range from zero to n, where $\sum_{i=1}^{s} n_i = n$, we can show that the probability of n points occurring in the union of s non-abutting intervals is

$$(11) \qquad {}_NC_n\left\{\frac{1 - \exp[-k\alpha_1]}{\exp[kt_1]} + \frac{1 - \exp[-k\alpha_2]}{\exp[k(t_1 + \alpha_1 + t_2)]}\right.$$

$$+ \cdots + \frac{1 - \exp[-ka_s]}{\exp[k\{\sum_{i=1}^{s}(t_i + a_i)\}]}\Bigg]^n .$$

$$\left[1 - \frac{1 - \exp[-ka_1]}{\exp[kt_1]} - \frac{1 - \exp[-ka_2]}{\exp[k(t_1 + a_1 + t_2)]} - \cdots\right]^{N-n} .$$

5. *Probability of n emissions during k closures of a counter.* It is possible to find the probability that n emissions has been emitted during k closures of a counter which follows Law 1. Let N be the number of untransmuted particles at $t = 0$, and let $t_1, t_2, t_3, \ldots, t_i, \ldots, t_k$ be the instances that the counter closes to register an emission. Hence at t_i the counter has registered i emissions. Let u be constant time that the counter is closed while counting an emission. It easily follows that the probability that n emissions have not been counted during two closures is

$$(12) \qquad \sum_{i=1}^{n} g(n - i, u, t_1, N - 1)\cdot g(i, u, t_2, N - n + i - 2).$$

Hence it is easily seen that the probability that n emissions have not been counted during k closures of the counter is

$$(13) \quad \Sigma g(n_1, u, t_1, N-1)\cdot g(n_2, u, t_2, N-n_1-2)\cdot g(n_3, u, t_3, N-n_1 - n_2 - 3)\cdots g(n_k, u, t_k, N - n + n_k - k),$$

where $n_1, n_2, \ldots, n_k$ can be any one of the integers $0, 1, 2, \ldots, n$, with $\sum_{i=1}^{K} n_i = n$ and the number of terms in the summation is $(n + k - 1)!/n!(k - 1)!$. Probabilities (12) and (13) can not be reduced to forms similar to (9) and (11) respectively.

REFERENCES

[1] L. Alaoglu and N. M. Smith, "Statistical theory of a scaling circuit," *Phys. Rev.*, Vol. 53, (1938), pp. 832-36.
[2] L. v. Bortkiewcz, *Die Radioaktive Strahlung als Gengenstand wahrscheinlichkeitstheoretischer Untersuchungen*, Berlin 1913.
[3] A. H. Copeland and F. Regan, "A postulational treatment of the Poisson law," *Annals of Math.*, Vol. 37, (1936), pp. 357-62.
[4] W. Feller, "On probability problems in the theory of counters," *Courant's Anniversary Volume*, Interscience Publishers, Inc., New York, (1948), pp. 105-15.
[5] T. C. Fry, *Probability and its Engineering Uses*, D. Van Nostrand Co., Inc., New York, 1929, pp. 235-37.
[6] B. V. Gnedenko, "On the theory of Geiger-Müller counters," (in Russian), *Jour. for Exp. and Theor. Phys.*, Vol. II (1941), pp. 101-6.
[7] J. D. Kurbatov and B. H. Mann, "Correction of G-M counter data,"

Phys. Rev., Vol. 68, (1945), pp. 40-53.
[8] L. Kosten, "On the frequency distribution of the number of discharges counted by a Geiger-Muller counter in a constant interval," Physica, Vol. 10 (1943), pp. 749-56.
[9] C. Levert and W. L. Scheen, "Probability fluctations of discharges in a Geiger-Müller counter produced by cosmic radiation," *Physica*, Vol. 10, (1940) pp. 225-38.
[10] H. Maier-Leibnitz, "Die Koinzidenzmethode and ihre Anwedung auf Kernphysikalische Probleme," *Physikal Zeitschrift*, Vol. 43, pp. 333-62.
[11] S. Malmquist, "A statistical Problem Connected with the Counting of Radioactive Particles," *Annals of Math. Stat.*, Vol. XVIII, (1947), pp. 255-64.
[12] A. E. Ruark and F. E. Brammer, "The efficiency of counters and counter circuits," *Phys. Rev.*, Vol. 52, (1937), pp. 322-24.

St. John's University, Collegeville, Minn.

St. Louis University, St. Louis, Mo.

(*Continued from back of Table of Contents.*)

L. J. Adams was born in Morgan City, Louisiana, on January 10, 1907. He was graduated from Tulane University with the B.S. degree in 1925, and from the University of Southern California with the M.A. degree in 1932, with additional graduate work in mathematics at the University of California at Los Angeles. He is co-author of textbooks in Aircraft Analytic Geometry, Intermediate Algebra, Commercial Algebra, and Mathematics of Finance. His special interests are in applied mathematics and the teaching of mathematics.

Don Lebell was born in Los Angeles, California, in 1926, and is a graduate of the University of California (B.A.S. '46; B.S. '47; M.S. '49). Since 1946 he has been engaged in teaching and research in the Department of Engineering, University of California, Los Angeles; and since 1949 he has been in charge of the University's differential analyser. Mr. Lebell is interested principally in mathematical techniques for the analysis of physical systems.

Melvin Dresher, Research Mathematician, RAND Corporation, was born in Poland in 1911. He attended Lehigh University (B.S. '33) and Yale University (Ph.D. '37). He taught at Michigan State College and Catholic University. During the war he served as a research mathematician with the War Production Board and as a mathematical physicist with the Allegany Ballistics Laboratory. Dr. Dresher has contributed research papers on group theory, game theory, and applied mathematics. Currently he is interested in the application of game theory to military and economic problems. (An article by Dr. Dresher on this subject appeared in the November-December issue, 1951).

John M. Howell, Mathematics Instructor, Los Angeles City College, was born in Pennsylvania in 1910. He attended L.A.C.C. and U.C.L.A. (B.A., '37, M.A. '47). He was with Northrop Aircraft from 1941 to 1946 and has been at L.A.C.C. since 1946. A past chariman of the Los Angeles section of the American Society for Quality Control, he was active in the formation of the local and national society. Mr. Howell has done consulting work in this field and has designed a slide rule for quality control calculations. (An article on quality control by Mr. Howell appeared in the January-February issue).

Milo W. Weaver is a graduate of the University of Texas (B.A. '35, M.A. '50). After teaching mathematics and science in the public schools of Texas for ten years, he returned to the University as Instructor of Extension Mathematics in 1945. A student of Professor Vandiver, Mr. Weaver is interested in abstract algebra, particularly in semigroups. (An article on this subject by Mr. Weaver appeared in the January-February issue).

SOME RELATIONS INVOLVING SPECIAL FUNCTIONS

L. I. Deverall and C. J. Thorne

Introduction

The Laplace transform is used to establish some identities involving the special functions of mathematical physics. The results are restricted to those which can be found without the explicit use of contour integration. The following definitions and elementary results are needed:

$$(1) \qquad (x + y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i$$

where the symbol $\binom{n}{i} = n!/(n - i)!\,i!$ denotes the usual binomial coefficient. From the formula for a finite geometric series with common ratio $(1/x)$, the following relation is easily established.

$$(2) \qquad \sum_{i=0}^{n} (1/x^{i+1}) = [1/(x - 1)] - [1/x^{n+1}(x - 1)].$$

The Laplace Transform

The Laplace transform of a function $F(t)$ is the function $f(s)$ obtained by the integral operation defined as follows

$$(3) \qquad L[F(t)] = \int_0^\infty e^{-st} F(t)\,dt = f(s).$$

For a discussion of the Laplace transform and its operational properties as well as sufficient conditions on $F(t)$ in order that its transform may exist, see Churchill (1). The notation and definitions of Churchill will be used throughout. We note here for later use the convolution property of the transform [see Churchill (1), p. 36],

$$(4) \qquad L^{-1}[f(s)g(s)] = \int_0^t F(\tau)G(t - \tau)\,d\tau,$$

where L^{-1} denotes the inverse Laplace transform. The LaGuerre and Hermite polynomials are defined by the following relations:

La Guerre Polynomial

$$L_n(t) = \frac{1}{n!} e^t \frac{d^n}{dt^n} (t^n e^{-t}). \qquad (n = 0,1,2,\ldots)$$

(5)

Hermite polynomial

$$H_n(t) = e^{t^2} \frac{d^n}{dt^n} (e^{-t^2}). \qquad (n = 0,1,2,\ldots)$$

We shall use the following notation for Bessel functions. For a discussion of the properties of Bessel functions, see for example, Watson (2).

$J_\beta(t)$ – Bessel function of the first kind of index β.

$I_\beta(t)$ – Modified Bessel function of first kind of index β.

We shall need the following table of Laplace transforms.

	$f(s)$	$F(t)$
(6)	$1/(s-k)$	e^{kt}
(7)	$(1/s^n)$, $(n=1,2,3,\ldots)$	$t^{n-1}/(n-1)!$
(8)	$s^{-(2n+1)/2}$, $(n=1,2,3,\ldots)$	$2^n t^{(2n-1)/2}/1\cdot3\cdot5\cdot\ldots(2n-1)\sqrt{\pi}$
(9)	$\Gamma(k)/s^k$, $(k>0)$	t^{k-1}
(10)	$(1/s)\left[\dfrac{s-1}{s}\right]^n$	$L_n(t)$
(11)	$(1-s)^n/s^{(2n+1)/2}$	$n!H_{2n}(t^{1/2})/(2n)!(\pi t)^{1/2}$
(12)	$(1-s)^n/s^{(2n+3)/2}$	$-n!H_{2n+1}(t^{1/2})/(2n+1)!(\pi)^{1/2}$
(13)	$s^n/(s^2+a^2)^{n+1}$	$t^n \sin at/2^n a(n!)$
(14)	$1/(s^2+a^2)^k$, $(k>0)$	$\pi^{1/2}(t/2a)^{(2k-1)/2}J_{(2k-1)/2}(at)/\Gamma(k)$
(15)	$1/(s^2-a^2)^k$, $(k>0)$	$\pi^{1/2}(t/2a)^{(2k-1)/2}I_{(2k-1)/2}(at)/\Gamma(k)$
(16)	e^{-ks}/s^β, $(\beta>0)$	$0, (0<t<k)$
		$(t-k)^{\beta-1}/\Gamma(\beta)$, $(t>k)$
(17)	$e^{-k/s}/s^\beta$, $(\beta>0)$	$(t/k)^{(\beta-1)/2}J_{\beta-1}(2\sqrt{kt})$
(18)	$e^{k/s}/s^\beta$, $(\beta>0)$	$(t/k)^{(\beta-1)/2}I_{\beta-1}(2\sqrt{kt})$
(19)	$(\log s)/s^k$, $(k>0)$	$t^{k-1}\left[\dfrac{\Gamma'(k)}{[\Gamma(k)]^2}-\dfrac{\log t}{\Gamma(k)}\right]$

where $\Gamma(t)$ is the Gamma function. This table is taken from a much more extensive table of Churchill (1). It is possible to obtain more relations with a larger table of transforms, but the above table will be sufficient for our purposes.

By taking $x = s$, in equation (2), and multiplying the result by e^{-ks}/s^β, $(\beta>0)$, we obtain the following relation

$$(20)\quad \sum_{i=0}^{n} e^{-ks}/s^{i+\beta+1} = [e^{-ks}/s^\beta(s-1)] - [e^{-ks}/(s-1)s^{n+\beta+1}].$$

By use of transforms (16), (6), and relation (4) we may write the inverse transform of (20):

$$(21)\qquad e^{-t}\sum_{i=0}^{n}(t-k)^{i+\beta}/\Gamma(i+\beta+1)=$$

$$\int_0^t e^{-\tau}\left[\frac{(\tau-k)^{\beta-1}}{\Gamma(\beta)}-\frac{(\tau-k)^{n+\beta}}{\Gamma(n+\beta+1)}\right]d\tau,\qquad (t>k).$$

Two other relations may be derived by taking $x = s$ in equation (2), and then multiplying by $e^{-k/s}/s^{\beta}$, and $e^{k/s}/s^{\beta}$, respectively, where $(\beta > 0)$, and then taking the inverse transform of both sides of the resulting equations. [Transforms (17), (18), (6), and relation (4).]

$$(22)\qquad e^{-t}\sum_{i=0}^{n}(t/k)^{(i+\beta)/2}J_{i+\beta}[2(kt)^{1/2}]$$

$$=\int_0^t e^{-\tau}\{(\tau/k)^{(\beta-1)/2}J_{\beta-1}[2(k\tau)^{1/2}]$$

$$-(\tau/k)^{(\beta+n)/2}J_{\beta+n}[2(k\tau)^{1/2}]\}d\tau,$$

$$(23)\qquad e^{-t}\sum_{i=0}^{n}(t/k)^{(i+\beta)/2}I_{i+\beta}[2(kt)^{1/2}]$$

$$=\int_0^t e^{-\tau}\{(\tau/k)^{(\beta-1)/2}I_{\beta-1}[2(k\tau)^{1/2}]$$

$$-(\tau/k)^{(\beta+n)/2}I_{\beta+n}[2(k\tau)^{1/2}]\}d\tau.$$

With $x = s$ in equation (2), and multiplying by $(\log s)/s$, we have

$$(24)\qquad \sum_{i=0}^{n}\frac{\log s}{s^{i+2}}=\left[\frac{\log s}{s(s-1)}-\frac{\log s}{s^{n+2}(s-1)}\right].$$

The inverse transform of (24) may be found from transforms (19), (6), and relation (4).

$$(25)\qquad e^{-t}\sum_{i=0}^{n}t^{i+1}\left[\frac{\Gamma'(i+2)}{[\Gamma(i+2)]^2}-\frac{\log t}{\Gamma(i+2)}\right]=$$

$$\int_0^t e^{-\tau}\left\{[\Gamma(1)-\log\tau]-\tau^{n+1}\left[\frac{\Gamma'(n+2)}{(\Gamma(n+2))^2}-\frac{\log\tau}{\Gamma(n+2)}\right]\right\}d\tau.$$

We shall now consider some relations involving LaGuerre and Hermite polynomials. If in equation (1), we take $x = 1$, and $y = -[1 - (1/s)]$,

we get the result:

$$(26) \qquad 1/s^n = \sum_{i=0}^{n} \binom{n}{i} (-1)^i \left[\frac{s-1}{s}\right]^i .$$

By multiplying both sides of (26) by s^{-1} and then taking the inverse transform [transforms (7) and (10)], we get a result of Feldheim (3).

$$(27) \qquad (t^n/n!) = \sum_{i=0}^{n} \binom{n}{i} (-1)^i L_i(t).$$

In equation (1), if we take $x = 1$, and $y = -[1 - (1/s)]^2$, we get the identity

$$(28) \qquad (1/s^n)[2 - (1/s)]^n = \sum_{i=0}^{n} \binom{n}{i} (-1)^i \left[\frac{s-1}{s}\right]^{2i};$$

a second expansion and multiplication by s^{-1} yields

$$(29) \qquad \sum_{i=0}^{n} \binom{n}{i} 2^{n-i} (-1)\ (1/s^{n+i+1}) =$$

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^i \left[\frac{s-1}{s}\right]^{2i} (1/s).$$

and the inverse transform [transforms (7) and (10)] gives

$$(30) \qquad t^n \sum_{i=0}^{n} \binom{n}{i} 2^{n-i} \frac{(-1)^i t^i}{(n+i)!} = \sum_{i=0}^{n} \binom{n}{i} (-1)^i L_{2i}(t).$$

It is possible to obtain many expansions of the type (30); for example, expand $\left[1 + k\left[\frac{s-1}{s}\right]\right]^n (1/s)$, ($k$ const.), and then take the inverse transform of both sides of the resulting equation. We shall now consider some relations involving the Hermite polynomials. By taking $x = 1$, and $y = \left[\frac{1-s}{s}\right]$ in (1), we get

$$(31) \qquad 1/s^n = \sum_{i=0}^{n} \binom{n}{i} \left[\frac{1-s}{s}\right]^i$$

and multiplying this result by $s^{-1/2}$,

$$(32) \qquad 1/s^{(2n+1)/2} = \sum_{i=0}^{n} \binom{n}{i} (1-s)^i / s^{(2i+1)/2}$$

and the inverse transform is found by use of formulas (8) and (11).

$$(33)\qquad 2^n t^{(2n-1)/2}/1\cdot 3\cdot 5\ldots(2n-1)\sqrt{\pi} =$$

$$(1/\sqrt{\pi t}) + \sum_{i=1}^{n}\begin{pmatrix} n \\ i \end{pmatrix} i!\,H_{2i}(\sqrt{t})/(2i)!\sqrt{\pi t}.$$

Upon simplification this becomes

$$(34)\qquad 2^n t^n/1\cdot 3\cdot 5\ldots(2n-1) = 1 + \sum_{i=1}^{n}\begin{pmatrix} n \\ i \end{pmatrix} (i)!\,H_{2i}(\sqrt{t})/(2i)!$$

If we take equation (31) and multiply by $s^{-3/2}$, and take the inverse transform from equations (9) and (12),

$$(35)\qquad t^{(2n+1)/2}/\Gamma[(2n+3)/2] = 2\sqrt{t/\pi}$$

$$- \sum_{i=1}^{n}\begin{pmatrix} n \\ i \end{pmatrix} i!\,H_{2i+1}(\sqrt{t})/(2i+1)!\sqrt{\pi}.$$

We may establish the following identity from (1).

$$(36)\qquad (1-x^2)^n = \sum_{i=0}^{n}\begin{pmatrix} n \\ i \end{pmatrix}(-1)^i x^{2i}$$

then by letting $x = a(s^2+a^2)^{-1/2}$, and multiplying by $(s^2+a^2)^{-(n+1)}$, we get

$$(37)\qquad s^{2n}/(s^2+a^2)^{2n+1} = \sum_{i=0}\begin{pmatrix} n \\ i \end{pmatrix}(-1)^i a^{2i}/(s^2+a^2)^{n+i+1}$$

and by taking the inverse transform [transforms (13) and (14)], we get the following identity

$$(38)\qquad t^{2n}\sin at/2^{2n}a(2n)! =$$

$$\sum_{i=0}^{n}\begin{pmatrix} n \\ i \end{pmatrix}(-1)^i a^{2i}\sqrt{\pi}(t/2a)^{(2n+2i+1)/2}J_{(2n+2i+1)/2}(at)/\Gamma(n+i+1).$$

Many more relations could be worked out by using the foregoing methods. Also, by using transforms other than the Laplace transform, other identities could be established.

BIBLIOGRAPHY

1. R. V. Churchill, *Modern Operational Mathematics in Engineering*, New York, N.Y., 1944, pp. 4-5, 295-302.

2. G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge, 1922.

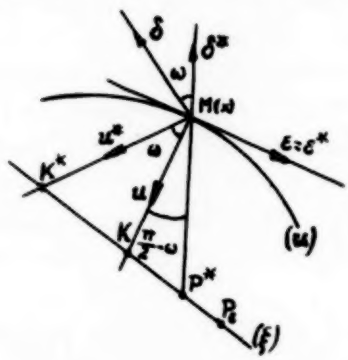
3. E. Feldheim, *Développements en série de polynomes d'Hermite et de LaGuerre a l'aide des transformations de Gauss et Hankel*, I et II, Nederl. Akad. Wetensch., Proc. 43, 1940, 224-248.

University of Utah

ON A GENERAL TRIHEDRON OF A CURVE

Alex. J. Yannopoulos

We consider, as a basic element of a space curve (u) with vectorial equation $x = x(s)$ (1), the sphere (P^*, P^*M) of center P^* on the axis of curvature (ξ) of (u) at his point M, so that $\sphericalangle (MK, MP^*) = \frac{\pi}{2} - \omega$ (MK the principal mormal of (u) at M) and radius P^*M. If ϵ, $\varkappa$, δ, are



the unit vectors of the Frenet's trihedron of (u) at M and ϵ^*, $\varkappa^*$, δ^* of a general three-perpendicular trihedron ($\epsilon^* = \epsilon$, $\delta^* // P^*M$, $\varkappa^* = \epsilon^* \times \delta^*$), we have $KP^* = \lambda . KP_\epsilon$, λ is a parameter and P_ϵ the center of osculating sphere of (u) at M. If $1/\rho$, $1/\tau$ are the curvature and torsion of (u) at M, we get:

$$KP_\epsilon = -\rho\tau\delta, \quad P^*M = x - (x + \rho\varkappa - \lambda\dot{\rho}\tau\delta) = -\rho\varkappa + \lambda\dot{\rho}\tau\delta,$$

$$(MP^*) = r^* = \sqrt{\rho^2 + (\lambda\dot{\rho}\tau)^2}, \quad \delta^* = (-\rho\varkappa + \lambda\dot{\rho}\tau\delta) / \sqrt{\rho^2 + (\lambda\dot{\rho}\tau)^2}$$

$$= -\rho\varkappa/r^* + \lambda\dot{\rho}\tau\delta/r^* = -\sin\omega\cdot\varkappa + \cos\omega\cdot\delta, \quad \sin\omega = \rho/r^*,$$

$$\cos\omega = \lambda\dot{\rho}\tau/r^*.$$

From $(MP^*) = r^* = \sqrt{\rho^2 + (\lambda\dot{\rho}\tau)^2}$, if $\lim \lambda = 1, \infty$, it is then respectively $\lim(MP^*) = \lim r^* = \sqrt{\rho^2 + (\dot{\rho}\tau)^2}$, ∞ and $M\cdot\epsilon^*\varkappa^*\delta^*$ is reduced to the Mayer's[1] or Frenet's trihedron.

Furthermore we have:

α′) $\dot{\epsilon}^* = \dot{\epsilon} = \varkappa/\rho = (\varkappa^*\cos\omega - \delta^*\sin\omega)/\rho = \varkappa^*/(\rho:\cos\omega) - \delta^*/(\rho:\sin\omega)$ or $\dot{\epsilon}^* = \varkappa^*/P^* - \delta/r^*$ (I), where $P^* = \rho:\cos\omega = \rho r^*:\lambda\dot{\rho}\tau$ (2).

If $\lim \lambda = \infty$, we have $\lim \omega = \theta$, $\lim P^* = \rho$ and for $\lim \lambda = 1$, is $\lim P^* = \rho r^* : \dot{\rho}\tau = P$ that is the radius of Mayer-curvature[(1)].

β') If we put $\delta^* = c_1\varepsilon^* + c_2\varkappa^*$, it is $\delta^*\varkappa = c_2 \cdot \cos\omega$, $\dot{\delta}^*\varepsilon = c_1$, $\dot{\delta}^*\varkappa = -\sin\omega = -\rho : r^*$, $\dot{\delta}^*\varkappa + \delta^*\dot{\varkappa} = -(\rho : r^*)^{\cdot}$, $\delta^*\dot{\varkappa} = \delta^* \cdot (-\varepsilon/\rho - \delta/\tau) = -\cos\omega/\tau$ and $c_2 \cos\omega = \cos\omega/\tau - (\rho/r^*)^{\cdot}$, $c_2 = 1/\tau - (\rho/r^*)^{\cdot} 1/\cos\omega$, $\dot{\delta}^*\varepsilon = -\delta^*\dot{\varepsilon} = -\delta^*\varkappa/\rho = \sin\omega/\rho = 1/r^*$, $c_1 = 1/r^*$ and thus $\dot{\delta}^* = \varepsilon^*/r^* + \varkappa^*/1 : [1/\tau - (\rho/r^*)^{\cdot} 1/\cos\omega]$ or $\dot{\delta}^* = \varepsilon^*/r^* + \varkappa^*/T^*$ (II), where $T^* = 1 : [1/\tau - (\rho/r^*)^{\cdot} 1/\cos\omega]$ (3).

If $\lim \lambda = \infty$ (it is $\lim \omega = 0$), we have $\lim (\rho/r^*)^{\cdot} = \lim[(1/r^*)^{\cdot}\rho + (1/r^*)\dot{\rho}] = \lim (1/r^*)^{\cdot}\rho] = \rho \lim(-\dot{r}^*/r^{*2}) = -\rho \lim[(\rho^2)^{\cdot} + \lambda^2[(\dot{\rho}\tau)^2]^{\cdot}] : 2r^*/r^{*2} = -\rho \lim[(\rho^2)^{\cdot} + \lambda^2[(\dot{\rho}\tau)^2]^{\cdot}]/2r^*[\rho^2 + \lambda^2(\dot{\rho}\tau)^2] = -\rho \lim[(\rho^2)^{\cdot}/\lambda^2 + [(\dot{\rho}\tau)^2]^{\cdot}]/2r^*[\rho^2/\lambda^2 + (\dot{\rho}\tau)^2] = 0$ and therefore $\lim T^* = \tau$. If $\lim \lambda = 1$, is $\lim T^* = T$ that is the radius of Mayer's torsion[(1)].

γ') $\varkappa^* = \varepsilon^* \ \delta^*$, $\dot{\varkappa}^* = (\varkappa^*/P^* - \delta^*/r^*) \times \delta^* + \varepsilon^* \times (\varepsilon^*/r^* + \varkappa^*/T^*)$ or $\dot{\varkappa}^* = -\varepsilon^*/P^* - \delta^*/T^*$ (III). From the general formulas (I), (II), (III), we obtain the Frenet's and Mayer's[(1)] formulas, for $\lim \lambda = \infty$ and $\lim \lambda = 1$.

(1) Alex. J. Yannopoulos: Study of Curves in reference to their Mayer (M_r) trihedron, Summary, Bull. Soc. Math. de Grèce, t. 25, p. 82-103.

University of Athens, Greece.

TEACHING OF MATHEMATICS

Edited by

Joseph Seidlin and C. N. Shuster

This department is devoted to the teaching of mathematics. Thus articles on methodology, exposition, curriculum, tests and measurements, and any other topic related to teaching, are invited. Papers on any subject in which you, *as a teacher*, are interested, or questions which you would like others to discuss, should be sent to Joseph Seidlin, Alfred University, Alfred, New York.

AN ALGORITHM ON DIVISIBILITY

N. A. Draim

I. Algorithm for finding the factors of an integer N by testing in succession the natural numbers until the smallest factor of N is reached. This is done by dividing the integer k into a dividend N_k which is smaller than N and which

(a) is derived from its predecessor N_{k-1} by the recurrent operation which forms the algorithm, and

(b) has the property that it is divisible by k if, and only if, the given number N is divisible by k.

When the first remainder zero is reached, k is the smallest factor of N other than unity, and the other factor, M_k, also appears. ($N = kM_k$). If $k = [\sqrt{N}]$ is passed without remainder zero, N is a prime number, and the first divisor, other than unity, is N.

Development:

$$\begin{array}{r|l|l} 2 & N & m_2 \\ \hline & 2M_2 & M_2 \\ \hline & & r_2 \end{array} \qquad \begin{aligned} M_2 &= N - m_2 \\ r_2 &= N - 2m_2 \\ N_3 &= M_2 + r_2 \\ M_3 &= M_2 - m_3 \\ N_4 &= M_3 + r_3 \\ &\text{etc.} \end{aligned}$$

$$\begin{array}{r|l|l} 3 & N_3 & m_3 \\ \hline & 3m_3 & M_3 \\ \hline & & r_3 \end{array}$$

$$\begin{array}{r|l|l} 4 & N_4 & m_4 \\ \hline & 4m_4 & M_4 \\ \hline & & r_4 \end{array}$$

$$\vdots$$

$$\begin{array}{r|l|l} k-1 & N_{k-1} & m_{k-1} \\ \hline & (k-1)m_{k-1} & M_{k-1} \\ \hline & & r_{k-1} \end{array}$$

$$\begin{array}{r|l|l} k & N_k & m_k \\ \hline & kN_k & M_k \\ \hline & & r_k \end{array}$$

$N = kM_k$, when $r_k = 0$.

Proof:

The validity of the algorithm is proved by the following relations:

(a) $$N_k = km_k - r_k$$

(b) $$N_k = (k-1)N - k\sum_{2}^{k-1} m_i$$

(c) $$N = kM_k - r_k$$

It is seen from these equations that $N_k \equiv (-N) \bmod k$, and from (c), that $N = kM_k$, if N is divisible by k.

The following characteristics of the algorithm are noted:

(a) Positive, negative, or least remainders may be used without affecting the results.

(b) The algorithm is non-terminating. Given $N = pq$, it will show first the factors p, q, and subsequently, the factors q, p.

(c) By the use of a skip process, even numbers may be eliminated as divisors, and the work appreciably shortened.

(d) By further modification of the skip process, all even and composite odd numbers may be eliminated, and the succession of prime numbers only, used as divisors.

(e) The number N, to be tested, may be broken into at any divisor, and the algorithm carried forward from that point.

The following examples are necessary and sufficient to illustrate the foregoing characteristics and to indicate the recurrent processes required. The examples may be generalized and proved inductively, but this is omitted in order to conserve space.

Example 1: Test 91, by the basic algorithm.

```
2|91 |_45                         46 = 91 - 45
  90   46                          1 = 91 - 2 x 45
       +1                         47 = 46 + 1
     3|47|_16                     30 = 46 - 16
       48  30                                 etc.
           -1
         4|29 |_7
           28  23
               +1
             5|24|_ 5
               25  18
                   -1
                 6|17 |_3
                   18  15
                       -1
                    7*|14 |_ 2
                       14   13*          91 = 7* x 13*
                             0
```

Example 2: Test 91, skipping even numbers as divisors.

```
 3|91 |30
   90  31
       +1
      5|32 |6
        30  19
            +2
          7*|21 |3
             21  13*
                  0
```

$$31 = 91 - 2 \times 30$$
$$19 = 31 - 2 \times 6$$
$$13 = 19 - 2 \times 3$$

$$91 = 7^{*} \times 13^{*}$$

Example 3: Continue the test, Example 2:

```
 7|21 |3
   21  13
        0
     9|13 |1
        9  11
           +4
       11|15 |1
          11   9
              +4
          13*|13 |1
              13  7*
                   0
```

$$91 = 13^{*} \times 7^{*}.$$

Example 4: Test 323, skipping all even and composite odd numbers as divisors.

```
 3|323 |108
   324  107
         -1
    5|106 |21
      105  65
           +1
        7|66 |9
          63  47
              +3
          11|97 |9
             99  29
                 -2
            13|27 |2
               26  25
                   +1
               17*|51 |3
                   51  19*
                        0
```

$$107 = 323 - 2 \times 108$$
$$65 = 107 - 2 \times 21$$
$$47 = 65 - 2 \times 9$$
$$97 = 47\frac{(11-7)}{2} + 3$$
$$29 = 47 - 2 \times 9$$
$$25 = 29 - 2 \times 2$$
$$51 = 25\left[\frac{17-13}{2}\right] + 1$$
$$19 = 25 - 2 \times 3$$

$$323 = 17^{*} \times 19^{*}$$

Example 5: Test 247, breaking in at 7, since 3 and 5 are obviously not divisors.

```
 7 | 247 |___
     245   35
           -2
       11 | 138 |13
            143  22
                 -5
              13* | 39 |3
                    39  19*
                         0
```

Change the *sign* of the first remainder (+2 to -2) $138 = (11 - 7) \times 35 - 2$

$22 = 35 - 13$

$39 = (13 - 11) \times 22 - 5$

$19 = 22 - 3.$

$$247 = 13^* \times 19^*.$$

American Embassy
Moscow, U.S.S.R.

These HEATH texts really go to work ...

CALCULUS

Tomlinson Fort

A distinguished book written with simplicity for the beginning student. The text contains many unusual features, and is pleasingly set in large type with clear figures. 576 p. *$4.75*

INTERMEDIATE ALGEBRA FOR COLLEGES

William L. Hart

A foundation for the student with inadequate preparation; a terminal course for those who need a substantial knowledge of algebraic techniques and computation. 276 p. text. *$3.00*

D. C. HEATH AND COMPANY

SALES OFFICES: NEW YORK CHICAGO SAN FRANCISCO ATLANTA DALLAS

HOME OFFICE: BOSTON

THE USE OF CONICS IN AIRPLANE DESIGN

L. J. Adams

In designing a new airplane the moment eventually arrives when it is necessary to describe the outer contours of the surface. The final shape is ordinarily the result of investigations and compromises by groups of engineers such as the preliminary designers, the aerodynamicists, the weight group, et cetera. In order to specify the contour it is convenient to use conics.[1,2]

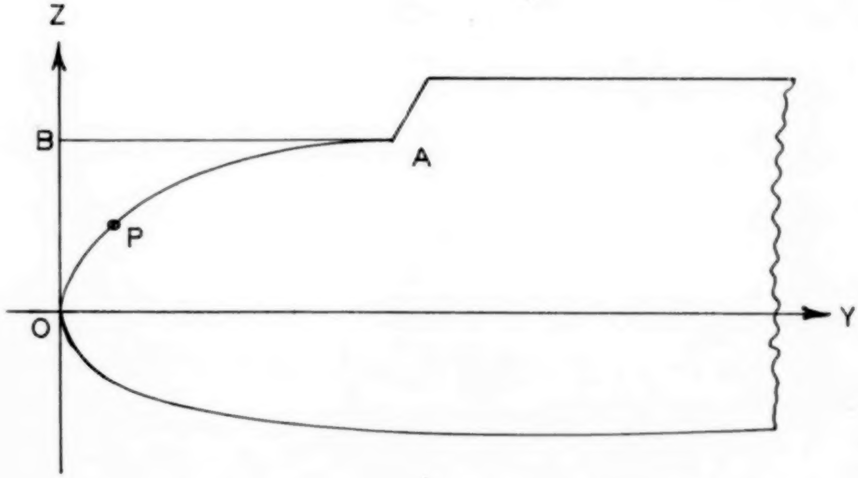


Fig. 1

To illustrate the problem and the method, consider the upper profile of the nose in Fig. 1.[3] This curve can be specified by the tangents AB and OB at A and O, respectively, and the fixed point P. This set of conditions is sufficient to determine a conic that meets the design requirements.

Using a conic has several advantages:

1. There is available a geometrical construction for the curve.
2. It is easy to write the equation of the curve, from which ordinates can be computed.
3. The entire well-developed theory and properties of conics is available for use in special situations and problems that may arise.

The geometrical construction is based on Pascal's famous theorem in projective geometry.

To construct additional points on the curve, follow the scheme illustrated in Fig. 2. The steps used in obtaining another point are

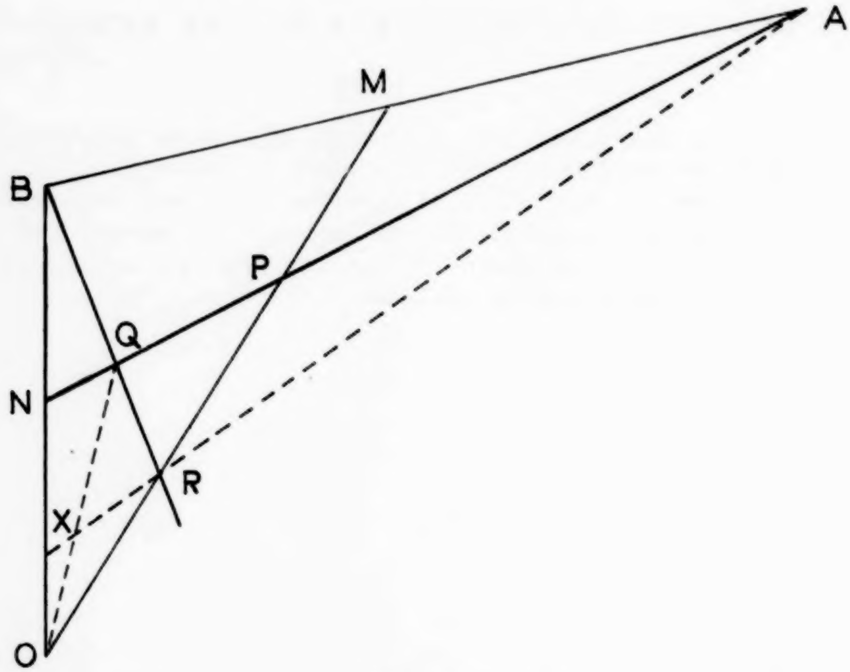


Fig. 2

1. Draw OM and AN through P. These are fixed lines throughout the construction.
2. From B draw a line in any desired position, intersecting AN in Q and OM in R. This line will yield a point on the curve for each position, as it rotates from BO to BA.
3. Draw OQ and AR (extended) intersecting in X. Point X is a point on the conic.
4. By varying the position of BR additional points on the conic can be found.

The entire profile of an airplane can be specified in this manner.

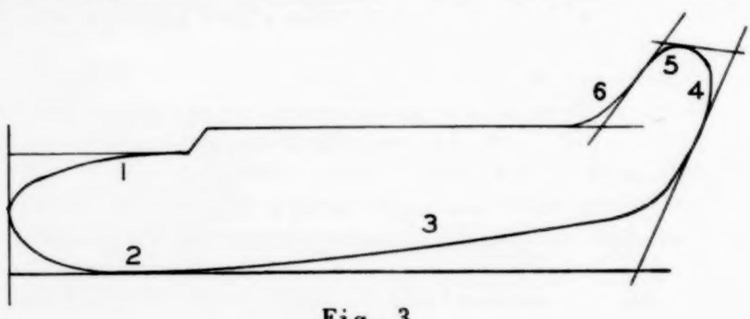


Fig. 3

In Fig. 3 a typical situation shows that an entire fuselage profile can be specified by six conics. The top in this figure is a straight line. In each of these six conics the designers would specify two

tangents (and the points of tangency) and a fixed point, as in Fig. 1.

Also, a typical cross-section of a fuselage may appear as in Fig. 4.

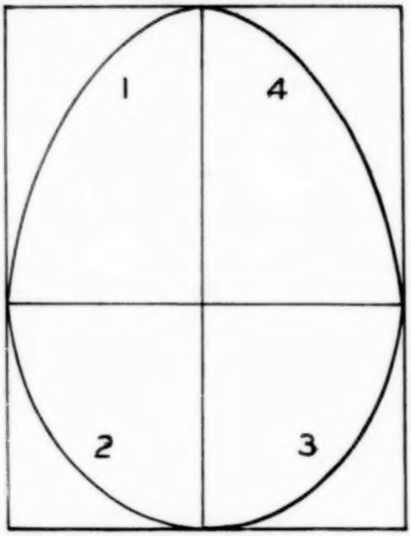


Fig. 4

Here the cross-section is specified by four conics, of which 1, 4 and 2, 3 are similar, because of symmetry.

In order to identify the construction in Fig. 2 with Pascal's theorem, we recall that a generalized concept of a hexagon is that a hexagon is a figure formed by 6 points of a plane, no three of which are collinear, and the line segments joining these six points in any order. Pascal's theorem states that the three pairs of opposite sides of a hexagon inscribed in a conic intersect in three points that lie on a straight line.

In Fig. 5, the hexagon 123456 is inscribed in a conic. The pairs of opposite sides are 12, 45; 23, 56; and 34, 61. Also 12 and 45 intersect at P; 23 and 56 intersect at Q; and 34 and 61 intersect at R. Pascal's theorem states that P, Q, R are collinear. The line through P, Q, R is sometimes called Pascal's line.

If we let 1, 2 coincide and 4, 5 coincide, then the lines 12 and 45 become tangents to the conic at points 1 (or 2) and 4 (or 5). Now if we suppose that point 3, say, is given but 6 is not, we can draw a Pascal line at random and use it to locate a sixth point on the conic.

In Fig. 6 line 1-2 is the tangent at 1, and line 4-5 is the tangent at 4. Also, point 3 is given. Then

$$\begin{aligned} &1\text{-}2 \text{ and } 4\text{-}5 \to P \\ &2\text{-}3 \text{ and } 5\text{-}X \to Q \\ &3\text{-}4 \text{ and } X\text{-}1 \to R \end{aligned}$$

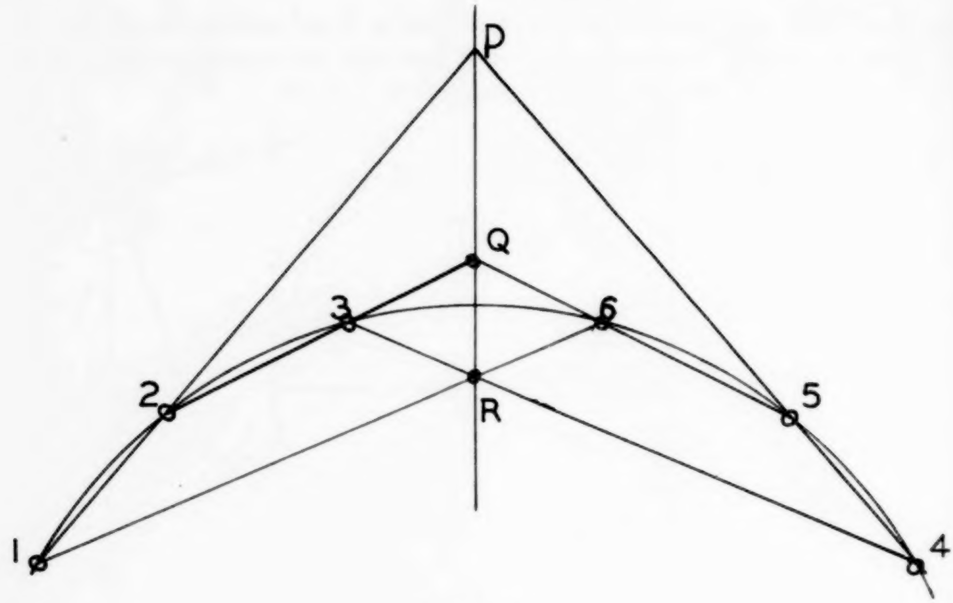


Fig. 5

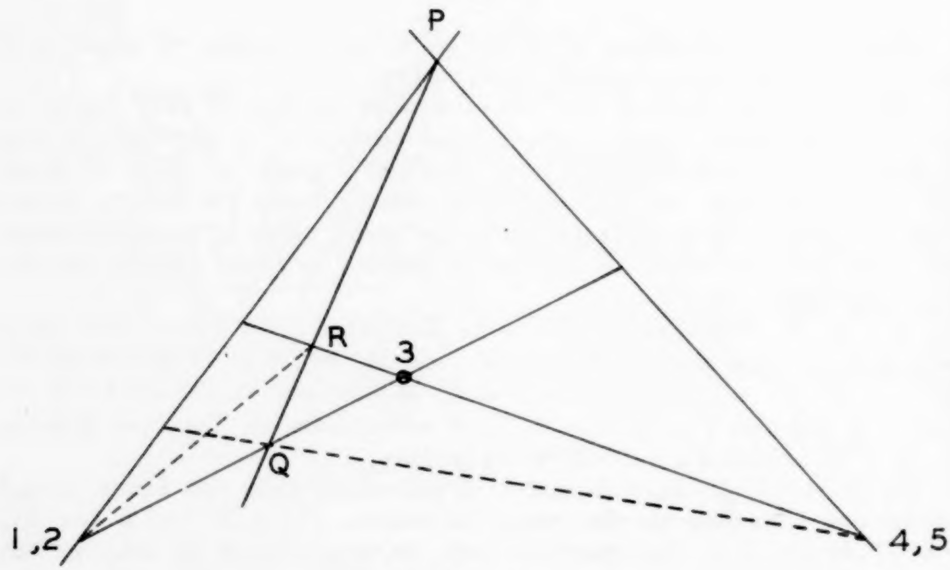


Fig. 6

Draw any line *PQR*. This is the Pascal line. Draw 2-3 and 5*Q*, also 3-4 and 1*R*. Draw 2-3; this intersects the Pascal line to give *Q*; draw 3-4; this intersects the Pascal line to give *R*. Draw 5*Q* and 1*R*; they intersect to yield *X*, a sixth point on the conic. Additional points *X* can be found by varying the position of the Pascal line.

Thus, the construction of Fig. 2 is identical with that of Fig. 6, and therefore it is an adaptation of Pascal's theorem. The position of the control point (P in Figs. 1 and 2, and 3 in Fig. 6) can be varied.

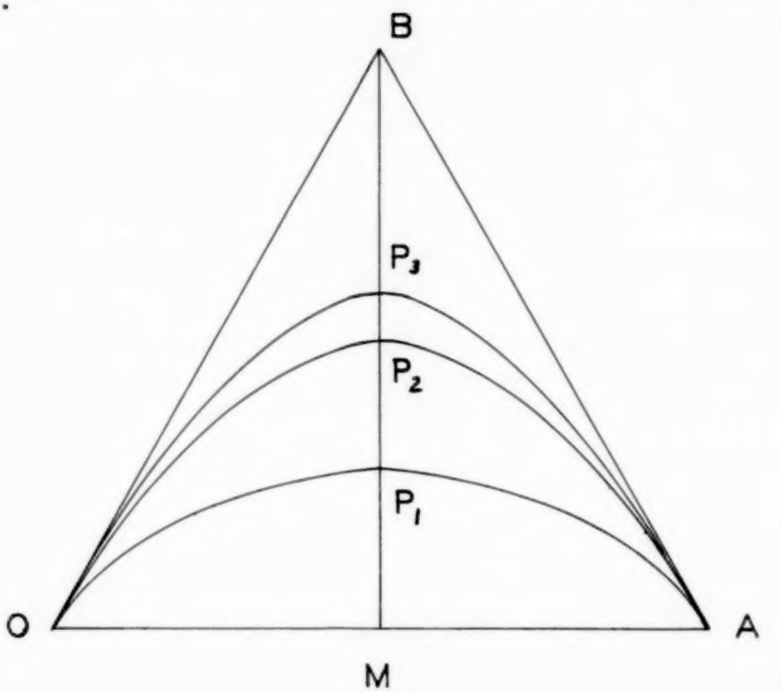


Fig. 7

If the tangents OB at O and AB at A are held fixed in Fig. 7, and the control point is allowed to take various positions on BM, where M is the mid-point of OA, it is easy to prove that the nature of the conic depends on the ratio $\rho = \frac{PM}{BM}$. This ratio is usually referred to as the ρ value of the conic. Then

$\rho < 1/2$ ellipse

$\rho = 1/2$ parabola

$\rho > 1/2$ hyperbola

To write the equation of the curve in Fig. 1, we proceed as follows.

Write the equation of the tangent AB, the tangent OB, and the chord OA. Although the tangent OB in Fig. 1 lies along the Y-axis, it is not necessarily in this position. See Fig. 8. Tabulate the equations:

$$AB: y - m_1 x - b_1 = 0$$

$$OB: y - m_2 x - b_2 = 0$$

$$OA: y - m_3 x = 0$$

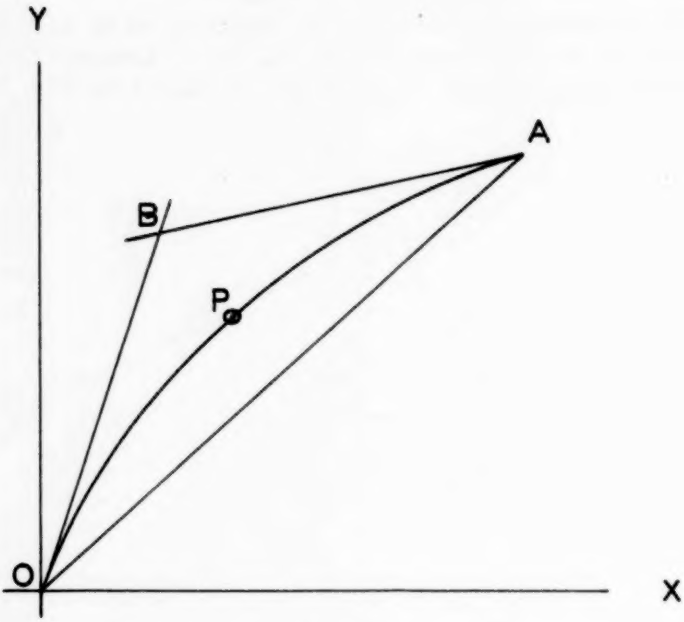


Fig. 8

then

$$(y - m_1x - b_1)(y_1 - m_2x - b_2) + k(y - m_3x)^2 = 0$$

represents a family of conics tangent to AB at A, and tangent to OB at O. To determine k and thus select from the family of conics the particular member that passes through $P(x_1, y_1)$, substitute the coordinates of P for x and y in the equation of the family and solve for k.

Solving the resulting equation for y in terms of x, we obtain an equation of the type

$$y = ax + b \pm \sqrt{cx^2 + dx + e}$$

Substituting values of x in the equation will yield ordinates.

Especially in the case of large airplanes it is desirable to use the equations of the profile curves rather than to lay out the profile by the geometrical construction. For manufacturing purposes, such as locating equipment, for example, the equations serve as well as a drawing, since the equations specify the exact nature and size of the contours. Ordinarily the cross-sections at various positions, such as the one illustrated in Fig. 4, are laid out geometrically by scribing with a knife on metal layout boards, although here, too, the equations may be used in lieu of or in conjunction with the layouts.

The student of synthetic and analytic projective geometry will

recognize the geometrical construction and the analytic equations as well known topics in those subjects. In fact, he may be surprised at seeing this very practical application of the theory of point conics.

One of the fundamental problems of lofting (laying out the contours of surface such as airplanes, boats, and ships) can be stated thus: given the shapes of the cross-sections of a surface at two places on the surface, to determine any number of intermediate cross-sections in such a way that the resulting surface will be "smooth".

The technique of doing this, as established in connection with ship lofting, was originally to "fair-by-eye"; that is, to "guess", the shapes of intermediate cross-sections and then to test the smoothness of the resulting surface by taking various sections and examining the contour curves of these test sections for smoothness. Other methods were developed, mainly in the automotive industry and partly in the aircraft industry, which were essentially based on ratio and proportion. With the advent of the use of conics new methods of lofting have been developed, and have proved very satisfactory.

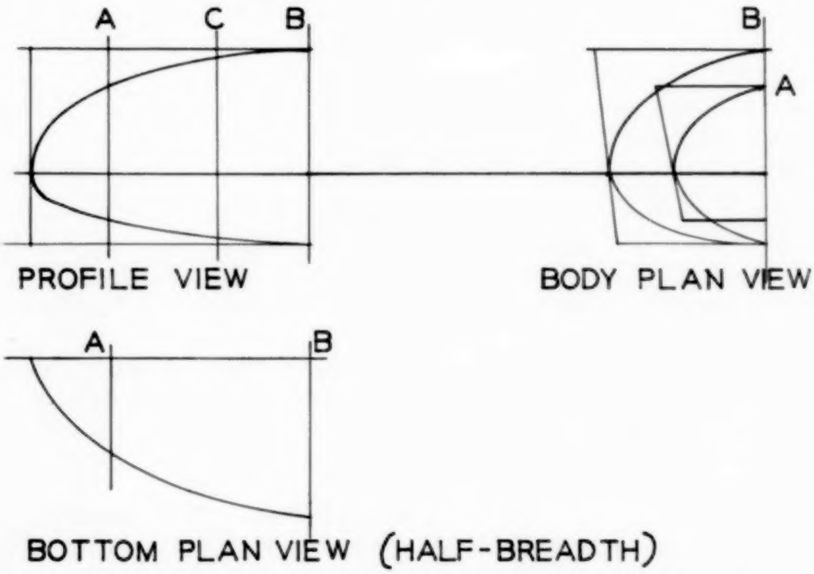


Fig. 9

In Fig. 9 the profile view shows the curve of intersection of the nose of a fuselage with the plane of symmetry of the airplane. This is similar to Fig. 1. The contour is described by two conics, an upper curve and a lower curve.

The body plan view shows the curves of cross-sections at fuselage stations. These curves may be specified by conics. The bottom plan view is the maximum half-breadth curve, which may be a conic.

The basic lofting problem is: given the contours at A and B, find

the contour at an intermediate station C. One method of solving this problem would be to proceed as follows: Use the profile view and bottom plan view to establish the vertical height and horizontal width of the contour at C. Let the tangents be parallel to the tangents at A and B. The only item left to determine is the position of the control point (or ρ value) of the upper and lower curves at C. To do this, one way is to draw a graph with the value of the upper curve at A as an ordinate at one end, and the ρ value at B at the other end and draw a conic between these two ordinates, reading the ρ value of C from this conic at the appropriate station. Fig. 10.

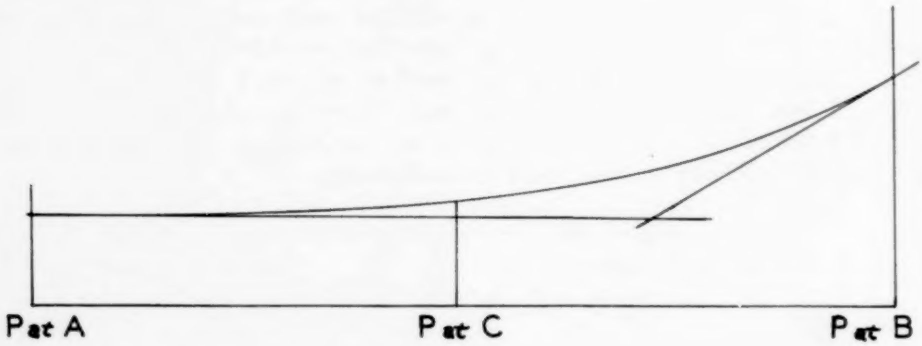


Fig. 10

The conic showing the ρ values is not unique. One advantage of this method over the method of using a straight line proportion is that the ρ values can be made to increase or decrease gradually away from A and toward B instead of having sharp increases or decreases at A and B.

Lofting by conics is also used in the design of the contours of automobiles and, perhaps to a lesser extent, in the design of boats.

REFERENCES

1. Apalategui and Adams. *Aircraft Analytic Geometry*. McGraw-Hill. 1944.
2. Liming, Roy. *Practical Analytic Geometry with Applications to Aircraft*. The Macmillan Co. 1944.
3. The drawings for this article were made by Richard Troon, student at Santa Monica City College.

Santa Monica City College

THE DIFFERENTIAL ANALYZER

Don Lebell

A preliminary general description of the computer field may serve to place the subject of this discussion in its proper context. The two basic classes of computers are digital and analog. While in some instances computers containing features common to both types threaten to upset this classification, their number is comparatively small and the analog-digital division remains distinct.

The I.B.M. equipment, desk calculators, the Eniac, Edvac, Binac, and Swac are all of the digital type. They are capable of performing high speed addition and subtraction of discrete numbers (digits) which are variously represented as electrical pulses, relay positions, electrostatic or magnetic "spots", or punched holes in tape or cards. Methods of numerical analysis can be applied which utilize this high speed arithmetic feature to perform the more complex operations of multiplication, division, integration and differentiation.

Analog computers however utilize the fact that mathematical operations or equations which describe the behavior of systems in widely different fields are frequently of the same mathematical form. For example, the distortions in the surface of a soap film can be described by the same mathematical equation which describes the states of stress and strain in the crank shaft of an engine. The explanations for this and other such apparent coincidences is that both equations are derived from the law of the conservation of mass and energy. Thus, soap films have been used in the computation of stresses in mechanical parts. The motion of electrons in an electrostatic field has been studied by observing the motion of marbles in the earth's gravitational field. The thermodynamics of a burning tree trunk have been studied by measuring the behavior of a specially designed electrical circuit.

In each of these examples, use of the analogy is completely dependent on the similarity of the equations of behavior of the analog and its prototype. The burning tree trunk's heat transfer equations (whose solution is desired) must correspond to the network equations of the electric circuit used, or a satisfactory analog does not exist.

There are a large variety of such special purpose analogs ranging from the common slide rule to very complicated electronic networks. In addition there are several more general purpose analog computers used to solve sets of algebraic equations, and finally there are the differential analyzers typified by the Ease, Reac, and Mechanical Differential Analyzer.

The differential analyzer is a general purpose analog computer consisting of individual components each of which performs a specific mathematical operation. Means are provided for recording the solutions, and for interconnecting the components in accordance with the equations

to be solved. Components are available for performing addition, subtraction, multiplication, division, integration and differentiation. Further, known functions can be plotted graphically and supplied to the problem by means of special input tables. The electronic differential analyzer performs these operations electronically with time varying voltages corresponding to the equations' variables. The mechanical analyzer performs these operations mechanically on the motion (angular position) of shafts within the machine. For example, to multiply X by 4, the shaft designated as the X variable is connected through a 4:1 gear ratio to an adjacent shaft which therefore becomes the quantity $4X$. The remaining terms in the equation are built up similarly until the entire equation has been "placed" on the machine. Then the shaft representing the independent variable is caused to rotate which actuates all other shafts (quantities) in the problem which are directly or indirectly functions of the independent variable.

Some of the problems which have been solved on the University of California differential analyzer located on the Los Angeles campus are listed as follows:

- Aeronautics and Aerodynamics
  - Aircraft flight and stability
  - Airfoil anti-icing
  - Landing gear design
  - Missile trajectories
  - Missile stability design
  - Stresses in aerodynamically loaded wings
- Electricity and Electronics
  - Pulse transformers
  - Electron accelerator studies
  - Magnetic amplifiers
  - Electron trajectories
- Oceanography
- Meteorology
- Servomechanisms
- Geophysics
- Fluid Flow
- Partial Differential Equations (two test problems)

In addition to their value as a tool in research and industry, many of these computers can be used to good advantage as teaching aids. Some of them provide excellent student projects.

University of California, Los Angeles

OPPORTUNITIES IN INSURANCE FOR THE MATHEMATICALLY TRAINED

Harwood Rosser

It is a common assumption that anyone working for an insurance company is a salesman. This is understandable, since insurance employees not in the selling end have relatively little contact with the public. To the average policyholder, the agent is not merely a representative of the Company; he is the Company. The Company encourages this attitude, even to the extent of arranging for him to deliver claim checks. This is good psychology on both sides.

The salesman is a necessary feature - I almost said "necessary evil" - of the insurance business, especially life insurance. It takes skill and persistence to persuade a man that he needs an insurance policy more than he does a new car, a television set, or some golf clubs. The history of savings bank life insurance, which is over-the-counter insurance, purchased on the initiative of the insured, is a sad commentary on human nature. For life insurance is the only form of insurance where a man will not suffer personally, as a result of being insufficiently insured, if the event insured against occurs.

We in the Home Office have our family quarrels with the agents. Yet we realize that we are all on the same team, even if they carry the ball most of the time. I doubt that I could carry it as well. However, my purpose is not to sell you an insurance policy, nor to proselyte you into selling insurance. Nor do the future salesmen in your classes need much mathematical background, except perhaps to figure their income tax returns.

Insurance of any kind is a risk-spreading device. It is not something for nothing. Some will get more than they paid in; others will get less, either directly, or through forfeited investment opportunities. Many people do not realize that life insurance shares this fundamental aspect of all insurance, and unjust complaints sometimes ensue.

The respect in which life insurance differs from all other forms is its long-term nature. Fire, automobile, and other casualty insurance policies are usually written for periods of one to three years. At the end of that time, the Company may change its rates, or refuse outright to renew. But it never has this opportunity on a life policy, once it is issued, if you keep up the payments, no matter what the state of your health.

This means that the guarantees in a life policy are not for a few years, but sometimes for as much as a century. They customarily cover the lifetimes of the insured and his ultimate beneficiary, who may not be born for another fifty years. To make estimates in advance over such a long period, discounting for interest, and taking account

of mortality and other contingencies, is a very complicated process. This is one reason why insurance companies employ actuaries.

You might suppose that eventually premiums and other figures would become stabilized. But fashions in insurance change too. For instance, in early England there were two classes of life insurance rates: one set for people who had had smallpox, and higher ones for those who had not. The automobile and the airplane have had their impact upon insurance contracts. So, perhaps, will the atom bomb.

Moreover, your policy may be participating. This means it receives dividends, which might better be called premium refunds. To keep the net cost, or premiums less dividends, in line with current experience is a continuous and elaborate process.

What does all this have to do, you ask, with the students in your classes? That must be answered in sections. A logical first section deals with the opportunities available to a high school graduate with a good record in mathematics.

There are a number of clerical positions, in a sizable life insurance company, that use mathematics in varying degrees. Most of these involve arithmetic only, usually performed on an adding machine or a calculator. However, in our Policyholders' Service Department, we utilize straight line interpolations and quadratic equations regularly. We sometimes try high school graduates, with a flair for mathematics, in these positions. Some of these graduates will already have encountered straight line interpolation in connection with logarithms. Even if they have not, it is much simpler to teach it to them if they understand the concept of similar triangles.

Double-entry bookkeeping is not beyond the grasp of a shrewd high school graduate, especially one who has had courses in it. But it is only fair to say that life insurance accounting is a system unto itself. Such a student might be able to qualify as an accountant. After that - well, what is an auditor but just a glorified accountant?

It is difficult to draw a sharp line for you between the positions available to those with high school training, and the ones open only to college graduates. An exceptional high school graduate would be eligible for some of the work listed hereafter. Our Personnel Department, like many others, gives an aptitude test to a prospective employee. This test includes arithmetic problems of varying difficulty. His performance on this, together with his educational background and the impression he makes in his interview, will determine the level of work we will offer him.

The insurance companies offer various opportunities for a college graduate. If he is trained in the mathematics of finance he may find a place in the investment or mortgage department.

To a man with a liking for figures, the business machine department would have considerable appeal. At present, most insurance companies

that can afford it have IBM installations. These utilize cards with holes punched in them. Once data are available on such cards, the machines will do many things much more rapidly and accurately than can be done by hand: not only arithmetical processes, but also sorting, listing, duplicating, and collating, among others. But these machines have one thing in common with the electronic "giant brains" that will undoubtedly replace them eventually. Both must be told what to do, and, usually, how to do it. Fortunately, unlike your students, they seldom talk back.

There is much virgin territory here. The ease with which these machines perform certain operations is revolutionizing our approach to many old problems. For instance, man devised the multiplication table as a short cut for addition. But now, where a whole series of multiplications must be performed, and the products added - a common problem in both insurance and statistics - a very ingenious method, called "digiting", replaces this multiplication by sorting and adding. This saves time, because these machines can add so much faster than they can multiply. In my humble opinion, these machines, and the people who can exploit their potentialities, have a tremendous future, not only in the more spectacular aspects, such as military research, but also in lessening the drudgery of the paper work necessary in any business office.

Then there is statistics. You may have heard a statistician defined as a man who can draw a straight line from an unwarranted assumption to a foregone conclusion. None the less, the statistical approach has value for an insurance company; for instance, in its annual financial statement. Certain asset and liability items are required purely for statement purposes, such as premiums due and unpaid on December 31. The Company is actually no more concerned with these than with premiums unpaid on any other date. However, a year-end figure must be obtained, either by inventory, or by estimating. The latter will save time, but must be reasonably accurate. If the estimate is based on samples, statistical formulas will indicate the probable accuracy.

This is a simple application of sampling theory. This theory can also be employed in mortality studies and other analyses. It is this theory that underlies "quality control", of which so much was heard during World War II.

Many of the job opportunities already mentioned are not confined to the insurance field. What I am going to describe now is a much more restricted line, which leans more heavily on mathematics than any of the preceding, I refer to the actuarial profession. That it is not crowded will be obvious to you from the fact that, in all of North America, there are less than a thousand fully qualified actuaries. That is, about one-thousandth of one per cent of the employed population fall into this category.

Most of these rare birds work for insurance companies. However, some are in consulting work, a few are in government employment, and you even find labor unions bidding for their services. The trend toward pensions, if continued, will increase the demand for non-company actuaries.

You are wondering what this esoteric individual does. He is prophet, salesman, detective, technician and business man. He must know both men and books; he is not just a bookworm. The presidents of about one-sixth of the life insurance companies have actuarial backgrounds.

As stated, the actuary is the company's prophet. His instrument is not a crystal ball, however, but such things as mortality tables.

The actuary must be a salesman also, at least of ideas. It is not enough for him to know the right answer. Unless he can get the rest of management to agree with him, he might almost as well be wrong. This often calls for a high order of persuasiveness, as well as the ability to explain technical processes and concepts in terms that can be understood by men without his mathematical background. This ability is valuable to him in other connections. As an example, the complaint letters that are really difficult to answer will usually be referred to an actuarially trained person. You will see that he needs a command of language, both written and spoken. This is recognized in the first actuarial examination.

I mentioned that the actuary is part detective. Perhaps I can illustrate. Statistics show that, age by age, married men live longer on the average than bachelors. If you were asked to explain why, you would probably say, after some deliberation, that married men have more regular hours, better meals, and somebody to watch over them. These are undoubtedly factors, which the actuary would realize. But he would not stop there. He would seek a possible deeper reason, and in this case he would find it. This more subtle factor is the tendency of a woman, in choosing a husband, to prefer a man who is a good physical specimen, with reasonable prospects of longevity. Her instinct warns her against becoming a widow with young children. In insurance circles, this is known as "selection".

Like this woman, a life insurance company attempts to exercise selection, by accepting only persons in good health at standard rates. Poorer risks are charged higher rates, if accepted at all. To classify applicants according to degree of risk, or to estimate their probable future lifetimes, is a highly involved process. This uses the results of medical examinations, among other things, and often requires the co-operation of the medical department. To calculate the corresponding extra premium is equally complicated.

These are some of the technical aspects of the actuary's work. Others have already been indicated. I may recall some of them to your minds by mentioning dividends, annual statements, and the long-term

nature of life insurance contracts. Moreover, his advise is often sought by other departments; for instance, in connection with amortization schedules.

Finally, and very deliberately, I labeled the actuary a business man. To achieve real success, he must be a business man first, and a mathematician second. He requires sound judgment and a breadth of viewpoint. The life insurance business is a very complex one, with, among others, mathematical, accounting, medical, legal and investment aspects. A man who rises in his company cannot be completely ignorant on any of these subjects. The syllabus of the actuarial examinations includes all these topics. While the earlier, or qualifying, examinations are primarily mathematical, the later ones take up the practical problems of running an insurance company.

The natural question now in your minds is: "How could one of my students become an actuary?" The answer is: "By passing the actuarial examinations." But this answer needs amplification.

The actuarial examinations are a set of examinations, given annually in the spring, under the auspices of a professional body, the Society of Actuaries. They are somewhat comparable to bar, medical, and C. P. A. examinations, which must be passed before an aspirant can practice a profession. There are also several differences. The actuarial examinations are nation-wide, so that there is no question of moving to another state and having to pass another set. Bar and medical examinations are all taken at one time. The actuarial examinations are a series of eight. The first five entitle the successful candidate to a junior degree, and the remaining three qualify him as a Fellow, or full-fledged actuary. But the maximum number that may be written in any one year is three, and examination regulations are such that at least four years are required. The average length of time needed is considerably higher.

There is no university in the world that gives a full actuarial course which prepares the student for all eight examinations. There are a few in this country, and in Canada, that offer courses through about the first four, sometimes granting degrees in actuarial science. Such degrees are not recognized by the Society of Actuaries, and their holders frequently fail the actuarial examinations. On the other hand, the actuarial profession is almost the only one in which a young man can earn a living while he is studying toward the coveted degree. Another advantage is that he is in a position to determine his probable success in the examinations before he graduates from college. As was inferred earlier, the first examination is a language test. After one year of calculus, the student has usually covered the material in the second examination. Hence he has two or three chances at these two, and perhaps the third one, before graduation.

The actuarial examinations are highly competitive, far beyond anything in most students' prior experience. I cannot possible overemphasize this. The knowledge required for the early examinations, except the first, is

intensive, rather than extensive. They are essentially speed tests. Enough material for a six-hour test is assembled, and then the candidate is given three hours. About one candidate in four passes Part 2, and one in five passes Part 3. Thereafter, the proportion rises somewhat.

It is admittedly difficult to get into this exclusive professional fraternity. It has been likened to a tightly held labor union, not entirely without justice. But for those who can achieve it, membership spells security and financial reward. Also, the Society's high standards protect, not only the members, but also their employers and the insuring public.

You will perceive the force of this more readily if you will consider the field of statistics, where there are fewer reliable yardsticks of competence. This is one reason why the science of statistics is often mistrusted. There are, of course, expert statisticians. But if you had to hire several, how would you go about selecting them?

It is neither necessary nor usual to complete the actuarial examinations before seeking employment. In fact, it is exceedingly rare. On the other hand, a college graduate looking for a job as an actuarial student would have a distinct advantage if he has already passed some of the examinations. He would have a further advantage, in that there would be fewer examinations left for him to study for outside of office hours.

Further information about the actuarial examinations, including sample questions, can be obtained from the Society of Actuaries, 208 S. LaSalle St., Chicago 4, Illinois.

CONCLUSION

I hardly need to suggest to this group of mathematics teachers that you urge your gifted students into more mathematics courses. But I certainly wish that my mathematics teachers had told me something about actuarial work several years sooner. For the younger you start, the better. The ability to pass competitive examinations usually fades rapidly with the years.

CURRENT PAPERS AND BOOKS

Edited by

H. V. Craig

This department will present comments on papers previously published in the MATHEMATICS MAGAZINE, lists of new books, and book reviews.

In order that errors may be corrected, results extended, and interesting aspects further illuminated, comments on published papers in all departments are invited.

Communications intended for this department should be sent in duplicate to H. V. Craig, Department of Applied Mathematics, University of Texas, Austin 12, Texas.

Infinite Matrices and Sequence Spaces. By Richard G. Cooke, Macmillan and Co. London. 1950. 347 pp.

It is the author's purpose to give an account of the properties and applications of infinite matrices as known up to the present time. No previous knowledge of finite or infinite matrices, or of algebra, is assumed. Because results are carefully stated, elucidated by many examples, and accompanied by many pertinent and precise references to sources of further information, it is expected that this book will turn out to be very useful both as a text for study and as a reference book. Three introductory chapters, pp. 1-57, are designed to accustom readers to manipulation of infinite matrices. The main part, pp. 58-223, treats general problems involving general transformations of the form $y_n = \sum_{k=0}^{\infty} a_{nk}x_k$. This is essentially unrelated to the other parts of the book and could be called a text on the theory of summability. Chapter 9, pp. 224-271, gives the elements of the Hilbert space of sequences with emphasis on orthonormal sets and Hilbert's binary form $\Sigma a_{pq}x_py_q$. Chapter 10, pp. 272-323 treats notions of convergence in several sequence spaces, notably Köthe-Toeplitz spaces.

The book differs from G. H. Hardy's *Divergent series* [Oxford, 1949, 396 pp.] in that it emphasizes general matrices a_{nk} instead of special classes such as those of Cesàro, Riesz, Nörlund, etc. There is surprisingly little overlapping in the two books. Unfortunately the author has, in a few fundamental definitions, failed to follow Hardy in adopting terminology which has now almost universally replaced older and sometimes misleading terminology. For example if $-1 < r < 1$, then, in usual terminology, the Cesàro method C_r is consistent with convergence but is not regular. But, by definitions of Cooke, page 58 implies that "C_r does not have the property of consistency" and page 113 implies that "C_r is regular". This situation calls for a discussion of history and terminology which is, at least in broad outlines, well known to many mathematicians.

Following Leibniz and other predecessors who toyed, in manners more philosophical than mathematical with the series $1 - 1 + 1 - 1 + \cdots$, Euler (1707-1783) developed and made effective use of many different methods for evaluating series. One of the great anomalies of mathematical terminology is the fact that the familiar power series method, a favorite weapon of Euler, should now be named after Abel (1802-1829) who conspired with Cauchy (1789-1857) to convince the world that no method other than the method of convergence can exist. It is thoroughly reasonable that, during the whole long process of recovery from the blight of Abel and Cauchy, say from 1880 to the present time, all methods A, B, C, $\cdots$ for evaluating series and sequences should be compared with the standard method known as convergence and, in so far as is possible, with each other. We now regard a method A as being an operator which, when applied to a sequence s_n (or series Σu_n) in the set A^* of sequences evaluable by the method, assigns a unique value s to the sequence so that $As_n = s$. The spirit of this attitude is neither improved nor impaired by writing $As_n = s$ in one or another of the forms $A\{s_n\} = s$, $(A)\{s_n\} = s$, $\{s_n\}(A) = s$, $\lim s_n = s(A)$, or $A - \lim s_n = s$; mathematical literature is full of such variants. Letting C denote convergence, we have $Cs_n = \lim s_n$ whenever s_n belongs to the set C^* of convergent sequences. Two methods A and B are now correctly called *consistent* (formerly mutually consistent) if no contradictions arise through application of the two methods, that is, if $As_n = Bs_n$ whenever both As_n and Bs_n exist. Two methods A and B are called *equivalent* (meaning equally effective and consistent) if $As_n = Bs_n$ whenever at least one of As_n and Bs_n exists. A method A is said to be at least as *effective* (or *strong*) as B if As_n exists whenever Bs_n exists, and is said to *include* B if $As_n = Bs_n$ whenever Bs_n exists. Thus A and B are equivalent if each includes the other. On account of the importance of the concepts, a method A is called *regular* if it includes convergence, and *conservative* (convergence-preserving) if it is at least as effective as convergence. It would be desirable to call a method A *subregular* if it is included by convergence, that is, if $\lim s_n = As_n$ whenever As_n exists; this would, for example, eliminate weird circumlocutions sometimes used in attempts to reveal the nature of Mercerian theorems. Without dwelling upon the origins of this basic, standard, sensible, and useful terminology, we remark that the term equivalent (équivalente) was used by Marcel Riesz in 1911 and that equivalence (Äquivalenz) and regular (regulär) were used by Issai Schur in 1913, all with the above meanings.

This commonplace history is relevant here because of a catastrophe with few if any parallels in mathematical terminology. Sometime in the interval from 1880 to 1913, a method A which we now call regular came to be known as a "consistent" method; not consistent with anything in particular, but just plain consistent. One could justify

this terminology if one could support an argument that it means "*A* is consistent with itself" or "*A* is consistent with good sense" or "*A* is consistent with convergence". The first meaning is futile, since each method *A* is obviously consistent with itself. The second alleged meaning may have been useful about 1905 in the psychological campaign to convince a wary public that there is respectibility in methods other than convergence, and could still have beneficial effects upon those historians who copy antique statements that the philosophy and work of Euler must be viewed with suspicion because he used divergent series. But certainly this meaning has no place in modern mathematics. The third meaning could easily be justified if use of the word consistent were appropriate; but it certainly is not. The following analogy should dispose of this matter. To say that the beliefs of man *A* are consistent with the beliefs of man *B* does not mean that "*A* believes everything that *B* believes, but not necessarily conversely"; it means that whenever *A* and *B* have beliefs that are comparable in the sense that they may be said to agree or to disagree, the two beliefs are found to agree. Since it was inevitable that the term *consistent* should creep into the theory with its true meaning, it is not surprising that, beginning perhaps with Schur in 1913, it has been found desirable to abandon the old term "consistent" in favor of the term "regular". The transition has been gradual. For example, the meaning of "consistency" in Hardy's celebrated paper "The second theorem of consistency ···" of 1915-1916 differs both from the meaning of 1905 and the present meaning; if Hardy had written the earlier paper when he wrote his book, the title would be "The second theorem of inclusion ···".

Two further comments, of which the first is less serious, should be made. The author systematically uses the word "efficient" when he means "effective"; to appreciate the difference in meaning of these two words, it is sufficient to observe that an inefficient method of killing a cat may be completely effective. Secondly, the Abel value of a series is called the "right value" of a series. This is an unfortunate cousin of the discarded concept that the value to which a convergent series converges is "right" and that everything else is "wrong", and it too should be discarded.

In view of all this, the reviewer hopes that the author will have an opportunity to incorporate improved terminology into a second edition of his excellent book.

Cornell University Ralph P. Agnew

The Human Use of Human Beings. (Cybernetics and Society). By Norbert Wiener. Houghton Mifflin Company, 1950. Pp. 242.

Intended as a sequel to *Cybernetics* (1950), this book is not strictly technical but is written for intelligent readers in any field. Its

burden is to clarify present economic, social, and political problems and to show the relationship between cybernetics and the solution of these problems. Cybernetics is explained carefully in terms of communication, selectivity ("listening" *for* something, in ordinary language), and the feedback principle. The author explains that "'feedback' means that behavior is scanned for its results, and that the success or failure of this result modifies future behavior". Consequently, feedback is a learning principle. It is a sort of communication between machines or between parts of a complicated machine. The author laments the fact that often technology enslaves human beings. He believes the modern phase of technology began with the exploitation of natural resources and often extends exploitation to human beings.

The author develops the thesis that secrecy in science is ineffective and basically a hindrance to science at home as well as elsewhere. He defends at some length the thesis that science and technology depend on the integrity of individuals and that ethics and morality are a necessary basis for the continuation of science. Ends, goals, or purposes can never be furnished by machines. Their proper use presupposes the integrity, evaluation, and choice of human beings. In this connection Wiener makes a distinction between "know how" and "know what". Science is rich in know how, but the people are poverty-stricken in know what (in the method of evaluating and selecting ends). In contrast to pure research applied science requires evaluation of ends.

The author confesses faith in democracy and in individualism and denounces several institutions and forms of government opposed to our American tradition, viz., Marxism and Roman Catholicism, which he believes have a common fascistic, totalitarian method.

Although this book is not highly systematic, it is exceedingly stimulating and worthwhile for anyone interested in the relationship between social, economic, religious, and political problems and technology.

University of Texas David L. Miller

PROBLEMS AND QUESTIONS

Edited by

C. W. Trigg, Los Angeles City College

Readers of this department are invited to submit for solution problems believed to be new and subject-matter questions that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by such information as will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink and twice the size desired for reproduction. Readers are invited to offer heuristic discussions in addition to formal solutions.

Send all communications for this department to *C. W. Trigg, Los Angeles City College, 855 N. Vermont Ave., Los Angeles 29, California.*

PROPOSALS

126. *Proposed by George Pate, Gordon Military College, Georgia.*

In firing a rifle at a target from a given distance, suppose that the probability of hitting the bull's-eye is 0.3. What is the smallest number of shots which must be fired in order that the probability of hitting the bull's-eye at least once will be 0.9?

127. *Proposed by M. R. Watson, San Fernando High School, California.*

AB is a diameter of circle (O). The externally tangent circles (O') and (O'') have their centers on AB and are also internally tangent to (O). Find the radius of (O''') which is tangent to (O), (O'), and (O'').

128. *Proposed by Victor Thébault, Tennie, Sarthe, France.*

In the division transformation $ABCD \times E + F = GHIJ$, no two letters represent the same digit. Identify the letters.

129. *Proposed by E. P. Starke, Rutgers University.*

Two equal ellipses are at first in coincidence. Then one of them is rotated about their common center through such an acute angle ϕ that each has just half its area in common with the other. Show that $\sin \phi = 2ab/(a^2 - b^2)$, where a, b are the lengths of the semi-axes.

130. *Proposed by Leo Moser, University of Alberta, Canada.*

Prove that if n great circles on a sphere intersect in more than two points then they will intersect in at least $2n$ points.

131. *Proposed by F. L. Miksa, Aurora, Ill.*

Between two parallel guides 5 feet apart two flat belts are moving. One, 3 feet wide is moving south at 2 ft./sec. The other, 2 feet wide,

is moving north at 5 ft./sec. There is a mark, A, on the guide next to the 3-feet belt, and another mark, B, on the other guide 4 feet north of a point directly opposite A. At what angle with the guide should a bug set out from A to reach B in the shortest time, if it can crawl no faster than 7 ft./sec.?

132. *Proposed by Samuel Skolnik, Los Angeles City College.*

Prove that if P_n is the nth prime (in the order of magnitude), then for any $\epsilon > 0$, there exists a positive integer N such that $n^{1+\epsilon} > P_n$ for all $n > N$.

SOLUTIONS

A Primality Test

92. [March 1951] *Proposed by P. A. Piza, San Juan, Puerto Rico.*

Prove that $2n + 1$ is a prime if $Q_r = \binom{n+r}{2r}/(2r+1)$ is an integer for $r = 1, 2, 3, \cdots, (n-1)$, and conversely.

Solution by E. P. Starke, Rutgers University. If $2n + 1$ is not a prime and $p(p < n)$ is any prime divisor of $2n + 1$, then we have

$$Q_{n-p} = \frac{(2n-p)!}{p!\,(2n-2p+1)!} = \frac{(2n-p)(2n-p-1)\cdots(2n-2p+2)}{p!}.$$

Since no one of the factors in the numerator is divisible by p, Q_{n-p} is not an integer.

If $2n + 1$ is a prime, then $n - r$ and $n + r + 1$ are relatively prime. Observing that $(n-r)\binom{n+r+1}{2r+1} = (n+r+1)\binom{n+r}{2r+1}$, we see that the integer $\binom{n+r}{2r+1}$ is divisible by $n - r$. Therefore, $Q_r = \binom{n+r}{2r}/(2r+1)$ $= \binom{n+r}{2r+1}/(n-r)$ is an integer for every $n - r = 1, 2, 3, \cdots, (n-1)$, that is, for $r = 1, 2, 3, \cdots, (n-1)$.

Isotomic Points on the Edges of a Tetrahedron

101. [May 1951] *Proposed by N. A. Court, University of Oklahoma.*

If E, F are two isotomic points on the edge BC (i.e. equidistant from the midpoint U of BC) of the tetrahedron $DABC$, and the lines AE, DE, AF, DF meet the circumsphere again in the points P, Q, R, S,

we have, both in magnitude and in sign,

$$(AE)(AP) + (DE)(DQ) + (AF)(AR) + (DF)(DS) = (BC)^2 + (AD)^2 + 4(UV)^2,$$

where V is the midpoint of DA.

Solution by Howard Eves, Champlain College. We have

$$(AE)(AP) = (AE)(AE + EP) = (AE)^2 + (AE)(EP) = (AE)^2 + (BE)(EC).$$

Similarly

$$(AF)(AR) = (AF)^2 + (AF)(FR) = (AF)^2 + (BE)(EC).$$

Therefore

$$(AE)(AP) + (AF)(AR) = (AE)^2 + (AF)^2 + 2(BE)(EC) = 2(AU)^2 + 2(EU)^2 + 2(BE)(EC).$$

Similarly

$$(DE)(DQ) + (DF)(DS) = 2(DU)^2 + 2(EU)^2 + 2(BE)(EC).$$

Finally

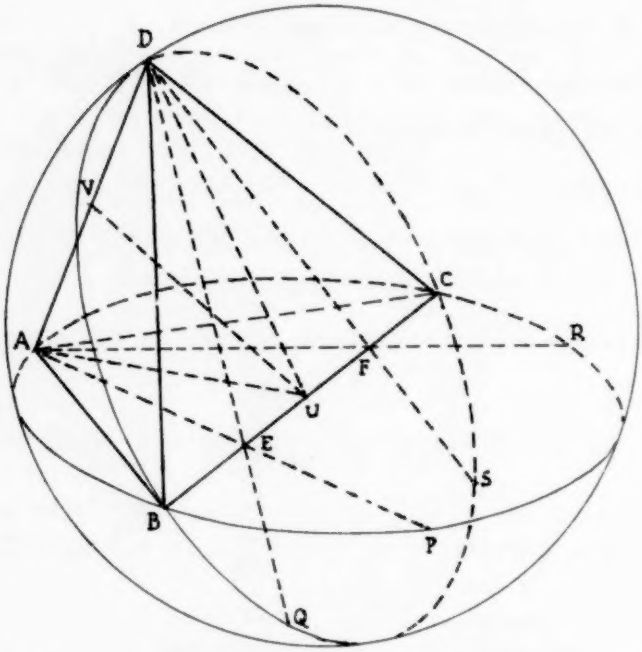
$$\begin{aligned}(AE)(AP) &+ (DE)(DQ) + (AF)(AR) + (DF)(DS)\\ &= 2(AU)^2 + 2(DU)^2 + 4(EU)^2 + 4(BE)(EC)\\ &= 2(AU)^2 + 2(DU)^2 + 4(EU)^2 + 4(BU - EU)(BU + EU)\\ &= 2(AU)^2 + 2(DU)^2 + 4(BU)^2\\ &= 4(UV)^2 + 4(AV)^2 + 4(BU)^2\\ &= 4(UV)^2 + (AD)^2 + (BC)^2.\end{aligned}$$

Also solved by *Leon Bankoff* (who supplied the figure), *Los Angeles, Calif.*; and *P. D. Thomas, Washington, D.C.*

Discussion by the proposer. The corresponding proposition in the plane was considered in *Educational Times, Reprints, N.S.*, **2**, 71, (1917), Q. 18243.

2. The sum $(BC)^2 + (AD)^2 + 4(UV)^2$ involves only the edge $BC = a$, the opposite edge $AD = d$ and the bimedian $UV = m_a$ of the tetrahedron $(T) = DABC$ relative to this pair of opposite edges. Hence, this sum is independent of the particular pair of isotomic points taken on the edge BC. Moreover, the same sum will be obtained, if the corresponding construction is applied to any pair of isotomic points E', F' taken on the edge AD. The sum of the two results for the two pairs of points E, F and E', F' is equal to

$$2(a^2 + d^2) + 8m_a^{\,2}.$$



If pairs of isotomic points are chosen arbitrarily on the other edges $CA = b$, $DB = e$, $AB = c$, $DC = f$ of (T), we obtain similar results for the two pairs of opposite edges CA, DB and AB, DC. Thus for the three pairs of opposite edges of (T) we obtain the sum

$$2[(a^2 + d^2) + (b^2 + e^2) + (c^2 + f^2)] + 8(m_a{}^2 + m_b{}^2 + m_c{}^2) = 4k^2,$$

where k^2 is the sum of the squares of the six edges of (T). [See the proposer's *Modern Pure Solid Geometry*, Macmillan (1935), page 56.] This result bears no trace of the manner in which the six pairs of isotomic points were chosen on the edges of (T).

3. A sphere (M) concentric with the circumsphere (O) of (T) determines pairs of isotomic points on the edges of (T). The construction considered applied to these six pairs of points will yield the result $4k^2$ which does not depend upon the radius of (M), and this result is obviously applicable to the circumsphere (O) itself.

The property may thus be attributed to a system of concentric spheres. A tetrahedron (T) is inscribed into one of the spheres of the system. The pairs of points considered on the edges of (T) may be determined by one sphere of the system, or the points on each edge may belong to a different sphere of the system. The result will be the same.

Analogous considerations may be applied to the plane.

Dissection of a Block

102. [May 1951] *Proposed by Leo Moser, Texas Technological College.*

What is the least number of plane cuts required to cut a block $a \times b \times c$ into abc unit cubes, if piling is permitted? Suggested by Q 12, **24**, 53, (Sept. 1950).

Solution by George Baker, Student, California Institute of Technology. Consider a line of length $r = 2^s + t$, where s and t are positive integers and $0 < t \leqq 2^s$. We desire to divide the line into r separate equal parts. With piling permitted, it is clear that after the first cut the problem confronting us is the division of the largest piece. The largest piece after the first cut must be equal to or greater than $2^{s-1} + [t + (0 \text{ or } 1)]/2$, the addition being made so that the fraction will be an integer. After s cuts the largest piece is equal to or greater than

$$1 + [t + (0 \text{ or } 1) + 2(0 \text{ or } 1) + 2^2(0 \text{ or } 1) + \cdots + 2^{s-1}(0 \text{ or } 1)]/2^s.$$

The fraction is now clearly equal to one, so only one more cut is required to complete the dissection, or a total of $s + 1$ cuts. Therefore, assuming the non-existence of the trivial case, a, b, or $c = 1$, we can write $a = 2^m + d$ $(0 < d \leqq 2^m)$, $b = 2^n + e$ $(0 < e \leqq 2^n)$, $c = 2^p + f$ $(0 < f \leqq 2^p)$. Then the minimum number of cuts is $m + n + p + 3$.

Also solved by the proposer, who writes the minimum number of cuts in the form

$$3 + [\log_2(a - 1)] + [\log_2(b - 1)] + [\log_2(c - 1)].$$

Two Independent Arrays

104. [May 1951] *Proposed by F. L. Miksa, Aurora, Illinois.*

Given the following two arrays of the 21 combinations of seven digits taken two at a time:

Array No. 1

1	2	3	4	5	6	7
25	16	12	17	14	15	13
34	35	46	23	27	24	26
67	47	57	56	36	37	45

Array No. 2

1	2	3	4	5	6	7
23	15	12	13	14	17	16
46	36	45	27	26	24	25
57	47	67	56	37	35	34

Show that no group transformation can be found which will transform Array No. 1 into Array No. 2.

Solution by the proposer. In Array No. 1 choose a number at random, say, 12. Since 12 is found in column 3, we record this information as 123. Now 23 is in column 4, so we extend our information to 1234. Next, 34 is in column 1, so we write 12341. Then 14 (the equivalent of 41) is in column 5, so we have 123415. After each addition to the growing number a check is put in the proper cell of the array. Continuing the process until there are two checks in each cell, we have a 44-digit cyclic code number, which *in every respect represents the array*. Thus Array No. 1 may be written as:

Code 1 – 44: 12341564317426714573654725162753246352137612.

If the process had been started with the number from any other cell of the array, the same fundamental cyclic code number would have been obtained.

Array No. 2 yields something entirely different, namely, two cyclic code numbers, one with 32 digits and the other with 14 digits. Thus Array No. 2 may be written as:

Code 2 – 32, 14: 12314536257167356415274265437512, 21347246176321.

Clearly, no group transformation on the seven digits will split the single code number of Array No. 1 into the two parts of the code number of Array No. 2. This completes the proof.

It will be observed that in each array, each column heading and the digits of the elements of that column constitute a distribution of the first seven positive digits. There are only five other such basic arrays possible, each having a different code number.

It is also possible to find all invariant transformations which will leave a given array invariant, except perhaps to permute the rows and columns of the array. In Array No. 1 if we start with 43 and place the code number under the code number for 12, we have

$$\begin{array}{cccccccccccccc} 1 & 2 & 3 & 4 & 1 & 5 & 6 & 4 & 3 & 1 & 7 & 4 & \ldots & 12 \\ 4 & 3 & 1 & 7 & 4 & 2 & 6 & 7 & 1 & 4 & 5 & 7 & \ldots & 43 \end{array}$$

Throughout, 1 appears below 3, 2 below 5, and so on, pointing to a transformation which will merely shift the code number laterally. Finding all such transformations is facilitated by placing the same code on two strips of narrow paper and matching the coincidences. Thus we find that Array No. 1 has a transformation group of order six: $\underline{1234567} \rightleftarrows 3521764 \rightleftarrows 2753461 \rightleftarrows 5472163 \rightleftarrows 7145362 \rightleftarrows 4317265$. Array No. 2 has a transformation group of order three:

$$\underline{1234567} \rightleftarrows 2746531 \rightleftarrows 7163542.$$

Minnie's Monetary Mix-up

105. [Sept. 1951] *Proposed by J. S. Cromelin, Clearing Industrial District, Chicago.*

Mrs. Minnie Moscowitz left home with no money. She stopped at the

bank to cash a check. Through an error which neither she nor the teller noticed, she was given as many dollars as her check read cents, and as many cents as the check read dollars. Next, Mrs. Moscowitz bought some stockings. She bought as many stockings as the check read dollars, and paid as much per stocking as the check read cents. She then counted her money and found she had just four times the amount of the check. How much was the teller short at the end of the day?

Solution by R. M. Swesnik, General American Oil Company of Texas, Dallas. If the check was for x dollars and y cents, then $100y + x - xy = 4(100x + y)$ or $96/x - 399/y = 1$ or $y = (3)(7)(19)/(96/x - 1)$. Since $y < 100$, then, $x < 20$, so the unique solution is $x = 12$, $y = 57$. Therefore the teller was short 57.12 - 12.57 or $44.55 at the end of the day, unless he made another mistake during the day.

Also solved by *Leon Bankoff, Los Angeles, Calif.; Louis Berkofsky, Roxbury, Mass.; H. E. Bowie, American International College; R. W. Byerly, New York, N.Y.; W. B. Carver, Cornell University; Monte Dernham, San Francisco, Calif.; F. F. Dorsey, South Orange, N.J.; A. L. Epstein, Geophysical Research Directorate, Cambridge, Mass.; L. R. Galebaugh, Lebanon, Pa.; B. K. Gold, Los Angeles City College; F. J. Howard, Huron, S.D.; E. S. Keeping, University of Alberta, Canada; H. R. Leifer, Pittsburgh, Pa.; F. L. Miksa, Aurora, Ill.; George Pate, Gordon Military College, Georgia; L. A. Ringenberg, Eastern Illinois State College; G. E. Williams, Student, Maryville College; and George Van Zwalvwenburg, Calvin College, Grand Rapids, Mich.*

Consecutive Integers Divisible by Perfect Squares

106. [Sept. 1951] *Proposed by E. P. Starke, Rutgers University.*

However large n may be, show that there exist n consecutive integers each of which is divisible by a perfect square.

I. *Solution by Ivan Niven's class in number theory, University of Oregon.* With n distinct primes $p_1, p_2, \cdots, p_n$ use the Chinese remainder theorem to obtain a solution x to the congruences $x + j \equiv 0 \pmod{p_j^k}$, $j = 1, 2, \cdots n$. Thus we have the more general result: for any positive integers n and k there exist n consecutive integers each of which is divisible by a non-trivial k-th power.

II. *Solution by E. de St. Q. Isaacsohn, Champion Reefs, Kolar Gold Fields, Mysore State, South India.* Euclid's Algorithm shows that if p and q have no common factor, then integers P and Q exist such that $P_p = Q_q + 1$. It is not difficult to extend this result to show that where the integers $p, q, r, s, \cdots$ have no common factor, then integers $P, Q, R, S, \cdots$ exist such that:

$$P_p = Q_q + 1 = Rr + 2 = Ss + 3 = \cdots.$$

An immediate result of this is that if we choose the numbers p, q, r, s, $\cdots$ as the squares of different primes, the proof of the proposition follows.

By this approach we are at liberty to choose, among other alternatives, the kth powers of different primes, and thus to extend the original proposition to say: "However large n and k may be, we are able to find n consecutive numbers each of which is divisible by a perfect kth power."

III. *Solution by S. B. Akers, Jr., U. S. Coast Guard Headquarters, Washington, D.C.* For $n = 1$, any perfect square will suffice. We shall assume that the proposition holds for n integers and then show that it will also hold for $n + 1$ integers. Let the n integers be a_1, a_2, $\cdots$, a_n and the perfect square divisors be p_1, p_2, $\cdots$, p_n, respectively. Let the least common multiple of p_1, p_2, $\cdots$, p_n be L, and let $A = a_{n+1}(L+2)L$. Consider the $n + 1$ consecutive integers, $A + a_1$, $A + a_2$, $\cdots$, $A + a_n$, $A + a_{n+1}$. The first n of these integers are divisible by p_1, p_2, $\cdots$, p_n, respectively, and $A + a_{n+1} = a_{n+1}[(L + 2)L + 1] = a_{n+1}(L + 1)^2$. This completes the induction.

Similarly, it may be shown that for any m and n there exist n consecutive integers each divisible by a factor of the form a^m.

Also solved by *W. B. Carver, Cornell University; F. L. Miksa, Aurora, Ill.; and the proposer*, who observed that for $n = 5$, the smallest set seems to be 844, $\cdots$, 848. For the squares of the primes 2, 3, 5, 7, 11, the smallest set is 1308248, $\cdots$, 1308252.

Parallel Chords in Two Intersecting Circles

107. [Sept. 1951] *Proposed by R. E. Horton, Lackland Air Force Base, Texas.*

Given circle C_1 intersecting circle C_2 in points R and T. From any point Q, lines QR and QT are drawn intersecting circle C_1 at A_1 and B_1, respectively, and circle C_2 at A_2 and B_2, respectively. Prove that A_1B_1 is parallel to A_2B_2.

I. *Solution by Perry Seagle, Nancy Stanton, and Mary Whitaker, students in S. W. Hahn's College Geometry class, Winthrop College, South Carolina.* If Q lies on the common chord RT, then the A's coincide, as do the B's. In this case, A_1B_1 and A_2B_2 are identical.

Now the product of the segments into which a point P divides a variable chord drawn from P to a given circle is constant. Also, from a point outside a circle the product of a secant and its external segment is constant. By these elementary theorems, if Q is not on RT, then $(QR)(QA_1) = (QT)(QB_1)$ and $(QR)(QA_2) = (QT)(QB_2)$. Hence, $(QA_1)/(QA_2) = (QB_1)/(QB_2)$ and A_1B_1 is parallel to A_2B_2.

II. *Solution by H. R. Leifer, Veterans Administration, Pittsburgh, Pa. Case 1.* If Q does not lie within C_1 or C_2, then A_1RTB_1 and A_2RTB_2 are inscribed quadrilaterals. Hence in C_1, angle A_1B_1T + angle $A_1RT = 180°$. In C_2, angle A_2B_2T + angle $A_2RT = 180°$. But angle A_1RT + angle $A_2RT = 180°$. Therefore, angle A_1B_1T + angle $A_2B_2T = 180°$ and A_1B_1 is parallel to A_2B_2.

Case 2. Q lies within C_1 but not within C_2. In C_1, angle A_1B_1T = angle A_1RT. In C_2, angle A_2RT + angle $A_2B_2T = 180°$. But angle A_1RT + angle $A_2RT = 180°$. Therefore, angle A_2B_2T = angle A_1B_1T and A_1B_1 is parallel to A_2B_2.

Also solved by *Leon Bankoff, Los Angeles, Calif.; Louis Berkofsky, Roxbury, Mass.; W. B. Carver, Cornell University; B. K. Gold, Los Angeles City College; A. Sisk, Maryville College, Tenn.; G. H. Thompson, Bell High School, Los Angeles, Calif.; G. Van Zwalvwenburg, Calvin College, Grand Rapids, Mich.; and the proposer.*

Interlocked Quartets of Spheres

108. [Sept. 1951] *Proposed by H. S. M. Coxeter, University of Toronto.*

Consider a set of four equal spheres, of radius $\sqrt{2} + 1$, all touching one another, and another similar set, of radius $\sqrt{2} - 1$. Prove that the two sets can be so placed that each sphere of either set touches three of the other.

Solution by W. B. Carver, Cornell University. The centers of the larger spheres must lie at the vertices of a regular tetrahedron of edge $2\sqrt{2} + 2$, and the centers of the smaller spheres at the vertices of a regular tetrahedron of edge $2\sqrt{2} - 2$. We place the smaller tetrahedron with its center at the center of the larger tetrahedron and its vertices on the altitudes of the larger tetrahedron at a distance $(2\sqrt{6} - 2\sqrt{3})/3$ in from each face. Then the distance from a vertex of the smaller tetrahedron to the three nearer vertices of the larger tetrahedron is $2\sqrt{2}$; and as this is the sum of the radii of the larger and smaller spheres, it follows that each smaller sphere will touch three of the larger spheres.

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

"**Q 48**. [Nov. 1951] Mentally multiply 38 by 32." *G. R. Jaffray* calls attention to a method quicker than that given in the November-December 1951 issue. Since $[10a + b][10a + (10 - b)] = 100a(a + 1) + b(10 - b)$, we have immediately $100(3)(4) + (8)(2) = 1216$. **Q 35** [May 1951] is a special case of this relation, which requires that both ten's digits be the same and that the sum of the two unit's digits be 10. It should be noted that the method of **A 48** is also applicable to the product of any pair of numbers with an even sum. E.g., since $(18 + 24)/2 = 21$, we have $(18)(24) = (21 - 3)(21 + 3) = 441 - 9 = 432$.

Q 51. [Jan. 1952] *Dewey Duncan* observes that since $1/7 = 0.\overline{142857}$, the first period of the repeating decimal equivalent to $1/7^2$ could be gotten more quickly than by *J. M. Howell*'s novel method by simply dividing the repeated periods (7 of them) by 7 until the divisor is exact. That is,

$$\begin{array}{r|l} 7 & 0.142857\ 142857\ 142857\ 142857\ 142857\ 142857\ 142857. \\ \hline & 0.020408\ 163265\ 306122\ 448979\ 591836\ 734693\ 877551 \end{array}$$

Q 53. [Jan. 1952] This Quickie should have read:
"Solve for z: $x/(x + 1) + y/(y + 1) + z/(z + 1) = 5/2$."

Q 55. Mentally multiply 47 by 67. [*R. L. Goodstein in The Mathematical Gazette*, **29**, 72, (1945).]

Q 56. Can a jailer enter one corner of a square block of 16 cells, then go to the diagonally opposite corner, entering each cell exactly once? [*Howard D. Grossman in Scripta Mathematica*, **14**, 160, (June 1948).]

Q 57. Prove that the derivative of an even function is odd and vice versa. [*Submitted by M. S. Klamkin.*]

Q 58. A thin continuous belt is stretched taut around three pulleys each 2 feet in diameter. The distances between the centers of the pulleys are 6, 9, and 13 feet. What is the length of the belt? [*Harry Langman in Scripta Mathematica*, **15**, 93, (March 1949).]

Q 59. Across one corner of a rectangular room two 4-foot screens are placed in such a manner as to enclose the maximum floor space. Determine their positions. [*F. Hawthorne in National Mathematics Magazine*, **19**, 321-3, (March 1945).]

Q 60. Sum the series: $\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n}$. [*Submitted by Leo Moser.*]

ANSWERS

A 55. Since $[10a + b][10(10 - a) + b] = 100[a(10 - a) + b] + b^2$, we have immediately $(47)(67) = 100[(4)(6) + 7] + 7^2 = 3149$.

A 56. Think of the prison cells as alternately colored like chessboard squares. Note that diagonally opposite corners have the same color. Let a "step" be a horizontal or vertical motion from one cell to the next. It takes an even number of steps to go from one cell to another of the same color. But since the number of cells is even, it takes one less or an odd number of steps to go through all the squares from first to last. Hence, No!

A 57. Since $E(x) = E(-x)$, $dE(x)/dx = dE(-x)/dx = [dE(-x)/d(-x)][d(-x)/d(x)] = -dE(-x)/d(-x) =$ odd. Since $O(x) = -O(-x)$, $dO(x)/dx = -dO(-x)/dx = [-dO(-x)/d(-x)][d(-x)/dx] = dO(-x)/d(-x) =$ even.

A 58. The radii to the points of tangency of the belt form angles which are supplementary to the angles of the triangle formed by the lines of centers. Hence the curved parts of the belt total a complete circle. The length of the belt is therefore $6 + 9 + 13 + 2\pi \doteq 34.2832$ feet.

A 59. The proposed figure is a quadrangle having one right angle and such that the two sides which meet in the vertex opposite the right angle are equal. Four congruent quadrangles of this form may be fitted together to form an equilateral octagon. For a given length of side, the octagon of maximum area is a regular octagon. Hence, ¼ this octagon will be the quadrangle of maximum area, $8(\sqrt{2} + 1)$ sq. ft. Therefore, the screens should be placed so that each forms with one wall and the bisector of the right angle at the corner of the room, an isosceles triangle of which the screen is the base.

A 60. Differentiate both sides of $(1 + x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n$ and set $x = 1$ to obtain $n(2)^{n-1}$ for the sum.

MISCELLANEOUS NOTES

Edited by

Charles K. Robbins

Articles intended for this Department should be sent to Charles K. Robbins, Department of Mathematics, Purdue University, Lafayette, Indiana.

THE EARLY TRAINING OF MATHEMATICAL RESEARCH WORKERS

During the years 1948-50 we had the following problem in the Mathematics Department of the newly founded University College of the Gold Coast. We were anxious that our students should, in as few years as possible, become qualified research workers and lecturers. Our negro students were of first-rate ability, but naturally of limited background. They had learned to differentiate x^n from schoolmasters who could not differentiate $\log_e x$. The schools gave little skill in problem solving, the teaching tended to be mechanical, and naturally could give no hint of more advanced topics to come.

Every young mathematician today faces a similar problem. There are vast branches of advanced mathematics one wants to get on to, but how to do this quickly without sacrificing the essential elementary foundations.

The compulsory examination was not until the end of the second year. We decided to forget all about it in the first year and to concentrate on achieving a psychological revolution, to make the students independent investigators and give them a wide view of mathematics. The students responded keenly. In order to give complete freedom, no examination at all was held at the end of the first year – even though other departments held examinations. During the first year we dealt with the topics of the official syllabus, but these were presented to the students as themes for research, rather than as ready-made theorems. Every opportunity was seized to digress to related questions in higher mathematics and to show elementary problems as embodiments of general principles.

Our aim was to stimulate the mathematical qualities. As we saw it, a mathematician is intrigued by *pattern*: he seeks new patterns suggested by those he already knows (*generalisation*); he feels *significance* in certain patterns and strives to interpret these; as the subject grows in bulk he feels the need for *unification*, for a few broad results to replace countless particular ones. Above all, he is an independent enquirer, not a learner by rote.

These maxims, of course, can only help the student if they are made concrete by definite examples. For instance: two general points can be connected by one straight line, $y = mx + c$. The student is invited to generalise this, to experiment with fitting quadratics to 3 points and so on until he arrives at the fitting of polynomials of degree n

to $(n + 1)$ points. In this investigation another topic arises; he finds $(n + 1)$ conditions on $(n + 1)$ unknown coefficients - is this enough to settle the question? A gegenbeispiel is given - 3 incompatible equations in three unknowns. When are linear equations incompatible? He finds the condition, and thus meets determinants of order 1, 2 and 3. Now for their properties. The student agrees that three incompatible equations written in a different order are still incompatible. How does this show itself in his condition? He thus sees that determinants *must* have certain properties. Most textbooks are very unsatisfactory on this point. There is a method (due I believe to Kronecker) of actually proving all the properties of determinants from general considerations without calculations.

A sustained investigation of this kind helps manipulation and a sense of mathematical form much more than disconnected examples.

An elementary example of unification: we sum A.P.s and G.P.s. Are these the only series with elementary sums? Is there any class of summable series that contains them both? Of course the arithmo-geometric series is the answer. Incidentally - and this is no accident - any solution of a linear differential equation $\phi(D)y = 0$ is summable, and gives the same class. Also, if in the formula for the sum of a G.P. $a(r^n - 1)/(r - 1)$ one puts $r = 1 + x$ and compares powers of x, one obtains the formula for the sum of an A.P. and also a whole set of new (to the student) results.

Even an unsolved problem left as an irritant in the student's mind has considerable value. In projective geometry, for instance, the cross ratio $f(a,b,c,d)$ of four points has the remarkable property that any permutation of a,b,c,d gives a simple rational function of the original f. (The verification of this, and the discovery of the rational functions is of course an exercise for the student.) The student should ask, "Why does this happen?" "What other functions, if any, have this property?" One can point out the connection of the group generated by $1/x$ and $1 - x$ with the group of the equilateral triangle, thus giving a hint that group theory will help him to understand this problem.

History often suggests ideas. For instance, to determine $\int_0^1 (1-x^2)^n dx$ for $n = 0, 1, 2, 3, 4, 5$, as Wallis did about 1650, is a simple exercise. What does the student notice about the results? One may go on (as Euler did) to evaluate $\int_{-1}^{1} (1 - x)^m (1 + x)^n dx$, - a slight generalisation. One arrives at the Beta functions, with several empirically evident properties. It is possible to go on to the Gamma Function. As a student, I was exasperated by lecturers who suddenly said "Let us consider $\int_0^\infty x^n e^{-x} dx$" without giving any indication why

one considered that particular function and not some other (apparently) equally worthy function.

The greater part of scientific research work and much of a mathematician's own work is simply elementary algebra. The systematising of elementary algebra is important as well as interesting, and it leads very quickly to higher algebra.

For instance, inequalities are often proved by appealing to squares being positive; can every inequality be proved in this way? The answer is probably - Yes, if the inequality holds for all real values of the variables. A theorem of Hilbert is relevant.

Again, is there any systematic way of proving $g = 0$ given that $f_1 = 0$, $f_2 = 0$, ... $f_n = 0$, all the functions being polynomials in several variables? The theorem stating that there must be an identity $g^r = w_1 f_1 + w_2 f_2 + \cdots + w_n f_n$ is interesting. It means that all such problems can be solved in one line.

The systematic establishment of determinant theorems is also an interesting field of enquiry. A question, for example; if $|M| = xy$ when can one find matrices A and B such that $|A| = x$, $|B| = y$, $AB = M$?

I should like to conclude with some questions. Does any textbook exist giving algebraic results of striking types, with some indication to students that these should be generalised and investigated? Or showing from what higher fields these examples are drawn?

Supposing my students continue indefinitely doing what I have suggested, generalising the results of elementary algebra, can one expect that ultimately these generalisations will fuse together into an organic whole or into a small number of separate generalisations? How far has this question been investigated?

Finally, would any of your readers be interested to correspond on this topic - the systematization of elementary algebra, and its relation to more advanced branches of mathematics?

Canterbury College
Christchurch, New Zealand W. W. Sawyer

[Editorial Note. This department of the Magazine will be glad to receive comments on Mr. Sawyer's paper.]

Time Space Coordinates Get on the Air

We hear: A college professor walking across the campus asked a student "What time is it, please?" "Half past one", the student replied. Whereupon the professor commented to himself, "Then I'm going from lunch and not to it."

A BACKGROUND OF UNDERSTANDING FOR GENERAL MATHEMATICS

Recently in a casual conversation with a boy in the third grade, I asked, "How many is nine take away six?" He hesitated a bit and said, "Three." "That's fine," I replied. "How many is seven take away five?" "Two." I then asked him, "How many is 22 take away nine?" "That's a hard one. Wait, I'll get it." After a time, he said, "Fourteen." As I watched him, it was obvious that he began with 22 and counted back nine numbers to 14. I next asked him, "How many is 52 take away 24?" "Whew, you got me there - I think I can get it." His answer, after some thought, was 32. It is probable that the boy had never before been asked to do this sort of example. In the new situation he used the best insights which he had. The insights available from previous experience served him fairly well, but inaccurately. "How do you do that?", I asked him. With a smile, a slight hesitation, and perceptible embarrassment, he said, "I got my own way." This anecdote illustrates the crux of the instructional problem in the teaching of arithmetic. When teachers do not provide children with meaningful ways of understanding mathematics, the children get their own ways.

As one studies the methods of instruction which have appeared in arithmetic textbooks over a period of ninety years, he sees the conflict of two theories of learning in the teaching of arithmetic. From 1860 to 1900 there was considerable emphasis upon the deductive approach. Children were taught the rules and then they were shown the applications. The influence of the opponents of formal discipline, led by Thorndike, may be seen in the books written early in the century. These experimenters did much good in overthrowing the grip which formal discipline had upon education. Whenever a strongly entrenched belief is overthrown, the usual outcome is for the impact to carry people too far. In this instance instruction moved far in the direction of specificity, and drill received great emphasis.

Although John Dewey, a contemporary of Thorndike and the author of a book on the teaching of arithmetic published in 1895, was laying the foundation for the experience curriculum, he was not heard. These decades were those of the experimenters and not of the philosophers. Was it not the philosophers who had established methodology up until 1900? However, as the weakness of the doctrine of specificity became more and more apparent, the experience curriculum grew in popularity. When one doctrine is overthrown, I repeat, there is a tendency for the opposition doctrine to go too far. The experience curriculum provide opportunities for learning some things exceedingly well. It does not provide for the systematic teaching of arithmetic in the way which seems necessary. When people champion a movement there is a tendency to exclude other approaches. To debate a controversy, sharp issues must be drawn. When issues are drawn sharply, teachers and superintendents observe them as dichotomies and are often misled into following one point of view to the

exclusion of others. The authors of this early period wrote about incidental teaching versus drill as if the two were discrete.

John Dewey, who popularized the terms "how much" and "how many", provided a basis for resolving some of the controversies. The teaching of "how much", which is the measures, can be done unusually well in an experience program. It makes little or no difference whether children are taught the meaning of a foot before a yard, or before a quart. It may be more meaningful for children to learn the meaning of a quart before a gallon, even though the name of the quart comes from a quarter of a gallon. Logic is secondary to the psychological experience of children as they learn the meaning of the measures. In learning the number system, on the other hand, it is hardly conceivable that children whose first need of numbers may be division, should learn the division facts before they learn to count. The learning of the number system must be orderly and relatively sequential. An experience program provides motivation by showing children the social significance of mathematics, but it must be supplemented by a carefully planned sequence of instruction.

During the last decade, the controversy between incidental instruction and planned lessons has been well resolved. Those who once championed incidental instruction speak of "unstructured lessons in a planned curriculum." Those who championed the strictly planned developmental lesson approach accept freely any teaching of the measures and the numbers in an experience program if the experience increases the meanings of the children.

If the children are going to carry a high residual from one grade to the next, strong learnings must be established by meaningful teaching.

It is my purpose to illustrate some of these basic arithmetical concepts and to point out how teachers might teach them. Only three of the concepts are discussed in this article. The first is the concept of the decimal system. If children are going to arrive at a high school course in general mathematics with something more than a number of easily forgotten facts, their instruction should involve the learning of the decimal system from as early as the second grade. The teacher in grade two works with children to discover the relative value of two digits when placed side by side in a positional relationship. By the use of sticks or cards, and "pockets", she teaches the meaning of place value. Holding three bundles of ten ice cream spoons and two single spoons, she may ask a child to take 16 from her. In so doing the child must "decompose" (a word for the teacher's language only and not for the child's) one of the tens. This act of the child is an experience as truly as is measuring the number of quarts in a gallon or the length of a room. The experience can be written on the blackboard in good mathematical language. All this process is carried out before the child has any knowledge of such a term as "borrowing". The language eventually emerges from the experience.

In later grades, understanding of the decimal nature of the number system is carried further. Its structure is contrasted with the limited place value of the Roman system. The digits in numbers are rewritten to show the largest and smallest amounts possible, e.g., 397, 973, and 379. The concept is invoked time after time through the grades. In multiplication, children learn the use of zero to indicate the absence of quantity in a particular place and eliminate a troublesome source of error. The concept is also used in teaching positional sequence in decimal fractions and mixed numbers, in developing understanding of per cents and, eventually, of logarithms.

A second illustration is the teaching of the dividend-divisor-quotient relationship. Quackenbos' *Practical Arithmetic*[1], 70 years ago, expressed the relationship in this manner:

> "I. With a fixed divisor, multiplying the dividend by any number multiplies the quotient by that number, and dividing the dividend divides the quotient.
>
> "II. With a fixed dividend, multiplying the divisor by any number divides the quotient by that number, and dividing the divisor multiplies the quotient.
>
> "III. Multiplying and dividing both dividend and divisor by the same number does not change the quotient."

When taught in this way, i.e., by beginning with the rules and making the application, it appears to be a sixth or seventh grade level concept. To test the relative difficulty of such a concept and its language was the purpose of a study by Beatty.[2] She asked children in the first grade such questions as this one: "Bobby's mother brought a plate of cookies for us to eat. Somebody has suggested that we invite the second grade to share the cookies. What will happen to the number of cookies each of us may have?" With no difficulty with the concept, they replied that there won't be as many cookies for each. They knew from experience that when the dividend remains the same while the divisor is increased by the addition of another class, the quotient will be smaller. Simple concepts are often stated in forbidding language, but good teachers teach the concepts through experience and stimulate children to provide the language. In a fourth grade one child who knew the concept devised his language structure of it in this manner:

A student teacher's comment was, "That is the clearest statement of

[1]John D. Quackenbos. *Practical Arithmetic*. D. Appleton & Co., New York

[2]Beatty, Leslie S. *Mathematical Judgment of Children in Early Elementary Grades*. Unpublished Master's Thesis, Claremont Graduate School, 1945.

the relationship I have ever seen." Even adults will profit when the relationship is taught inductively.

Another teacher was skillfully drawing from a class the relative size of 1/2, 1/3, 1/4, and 1/5. With a flash of insight, one little girl screamed, "Kids, kids, don't let 'em fool you, those numbers go the other way." She had seen the dividend-divisor-quotient relationship, but the formal language of Quackenbos must be delayed several years for her.

The elementary teacher who knows mathematics well enough will utilize this concept as he teaches quotient estimation in Grades Four and Five. The concept is particularly useful in the teaching of decimal fractions. It serves in the teaching of the reduction of fractions and in changing fractions to a common denominator.

The nature of the decimal system and the dividend-divisor-quotient relationship are but two of several fundamental concepts which contribute much to children's facility with mathematics. Effective teaching leads children to develop the generalizations themselves, in their own language and out of their own insights. These are permanent.

Another example of the use of relationships may be seen in the teaching of multiplication. An experiment in the teaching of the multiplication and carry facts by first teaching relationships and following by time controlled drills produced rapid gains in three fifth grade groups. Experience preceded drill. The experience approach is just as important for number concepts as it is for teaching the measures. Children who are to arrive at high school general mathematics, or any mathematics, can be made "ready" through a balanced program of meaning and drill. Either alone is not enough. A carefully planned combination of the two, with meaning emerging first and drill used to fix it and to produce quick response, gives promise of substantial, and often dramatic, achievement in arithmetic.

If teachers do not give children sound and serviceable meanings upon which to construct a system of numbers, they surrender by default to the approach of the third grader who said, "I got my own way." Helpful as his way may be, good teaching should lead him to develop the better ways. Then children progress securely and rapidly. Their gains are lasting.

San Diego State College Richard Madden

A
Fo
Ar
In
Cu
Mi
Pr
Ou
in
ap